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Anisotropic mesh adaptation for finite element solution of Anisotropic Porous Medium Equation

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ABSTRACT

Anisotropic Porous Medium Equation (APME) is developed as an extension of the Porous Medium Equation (PME) for anisotropic porous media. A special analytical solution is derived for APME for time-independent diffusion. Anisotropic mesh adaptation for linear finite element solution of APME is discussed and numerical results for two dimensional examples are presented. The solution errors using anisotropic adaptive meshes show second order convergence.

1. Introduction

In this paper, we extend the porous medium equation (PME) to the Anisotropic Porous Medium Equation (APME) that takes into consideration the anisotropic physical properties of the porous media such as permeability. Then we study the linear finite element solution of APME. We consider the following problem

$$\begin{cases} u_t = \nabla \cdot (u^m \mathbb{D} \nabla u), & \text{in} \quad \Omega_T = \Omega \times (t_0, T] \\ u(\boldsymbol{x}, t) = 0, & \text{on} \quad \partial\Omega \times (t_0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \text{in} \quad \Omega \times \{t = t_0\} \end{cases} \tag{1}$$

where $u = u(\boldsymbol{x}, t)$ is a nonnegative scalar function, $m \geq 1$ is the physical parameter, $\Omega \subset \mathbb{R}^d$ is a connected polygonal or polyhedral domain of d-dimensional space, $t_0 \geq 0$ is the starting time, $T > 0$ is the end time, $u_0(\boldsymbol{x}) \geq 0$ is a given function. We assume that $\mathbb{D} = \mathbb{D}(\boldsymbol{x})$ is a general symmetric and strictly positive definite matrix-valued function on Ω_T that takes both isotropic and anisotropic diffusion as special cases. For simplicity, we consider only time-independent diffusion matrix $\mathbb{D}$ in this work. The principles can also be applied to the time-dependent situation with minor modifications.

Porous medium equation (PME) arises in many fields of science and engineering such as fluid flow in porous media, heat transfer or diffusion, image processing, and population dynamics [1]. The general form is given as

$$u_t = \nabla \cdot (\nabla u^{m+1}), \tag{2}$$

or in the modified form

$$u_t = \nabla \cdot (u^m \nabla u). \tag{3}$$

For gas flow in porous media, m is the heat capacity ratio, u represents the density, u^m is the pressure, and $-\nabla u^m$ is the velocity.

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Mathematically, the parameter m in (3) can take any real value. Specifically, when $m = 0$, the PME (2) or (3) reduces to the heat equation. When $m > 0$, the PME becomes a nonlinear evolution equation of parabolic type that has attracted interests of both theoretical and computational mathematicians. Particularly, when $m = 1$, the PME is called the Boussinesq's equation that models groundwater flow in a porous stratum.

The nonlinear term u^m in (3) induces the so-called nonlinear diffusion that brings up many challenges in the analysis of the PME. However, this nonlinear diffusion is not the physical property of the porous media such as permeability. In this paper, we generalize PME to Anisotropic Porous Medium Equation (APME) that takes into account the anisotropy of the physical property of the porous media. In the mean time, APME can also be viewed as a nonlinear anisotropic diffusion problem.

General PME (2) or (3) has been studied extensively both in theory [1–8] and numerical approximations [9–21]. In particular, the nonlinear diffusion term and the sharp gradient near the free boundary make it difficult to achieve high order convergence of the numerical solutions. For example, finite difference moving mesh method has been developed in [17,20] for PME in one-dimensional space and second order convergence has been observed; however, no result is provided for PME in 2D. Error estimates have been developed in [15] for finite volume discretization of PME in 2D, which shows only first order convergence. For finite element discretization using quasi-uniform meshes [9,12,22–24], the convergence is at most first order for $m = 1$ and decreases for larger m. For one dimensional PME, high order convergence rate was achieved on a uniform mesh by using a high order local discontinuous Galerkin finite element method [14].

On the other hand, adaptive meshes and moving meshes are great choices to improve computational efficiency and accuracy by concentrating more mesh elements in the regions where solution changes significantly. In particular, Ngo and Huang [21] have studied moving mesh finite element solution of PME and demonstrated the advantages of moving mesh over quasi-uniform meshes. Their results show second-order convergence of solution errors using moving meshes.

However, no result is currently available for APME. Interesting features of PME also appear in APME such as finite propagation, free boundaries and waiting time phenomenon. Moreover, with the anisotropy of the porous media, satisfaction of maximum principle becomes more challenging and special mesh adaptation is needed, see [25–27] and the references therein. This paper serves as a starting effort about APME and its numerical solutions. Anisotropic mesh adaptation technique is applied in the numerical computations to improve efficiency and accuracy. Different than moving mesh method that keeps the connectivity of the elements, our anisotropic mesh adaptation technique can change the connectivity as well as the number of elements as desired. Therefore, a coarse uniform mesh can be used as the initial mesh in our adaptation, while a fine initial mesh is usually needed for moving mesh method in order to capture the sharp change of solution on the initial free boundary.

The outline of this paper is as follows. In Section 2, the anisotropic porous medium equation (APME) is developed and the exact solution for a special case is discussed. Section 3 gives a brief summary of linear finite element solution of the APME, and Section 4 introduces the anisotropic mesh adaptation methods. Numerical examples are presented in Section 5 to show the different behavior between APME and PME and also demonstrate the advantages of adaptive anisotropic meshes over other meshes. Finally, some conclusions are drawn in Section 6.

2. Anisotropic Porous Medium Equation (APME)

Firstly, we derive the model for fluid flow through anisotropic porous media as follows. Let u be the density of the fluid. The flow is governed by the following three equations.

(I) Conservation of mass (continuity equation)

$$\varepsilon u_t = -\nabla \cdot (u\,\boldsymbol{v}), \tag{4}$$

where $\varepsilon \in (0, 1)$ is the porosity of the media and $\boldsymbol{v}$ is the velocity.

(II) Darcy's law in anisotropic porous media

$$\boldsymbol{v} = -\frac{1}{\mu}\mathbb{D}\nabla p, \tag{5}$$

where μ is the viscosity of the fluid, $\mathbb{D}$ is the permeability matrix of the porous media, and p is the pressure. In most cases, the porous media is anisotropic, thus $\mathbb{D}$ has different eigenvalues. If $\mathbb{D}$ varies with location, then it also represents heterogeneous media.

(III) The equation of state

$$p = p_0 u^m, \tag{6}$$

where p_0 is the reference pressure and $m \geq 1$ is the ratio of specific heats.