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An example of explicit implementation strategy and preconditioning for the high order edge finite elements applied to the time-harmonic Maxwell's equations

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ABSTRACT

In this paper we focus on high order finite element approximations of the electric field combined with suitable preconditioners, to solve the time-harmonic Maxwell's equations in waveguide configurations. The implementation of high order curl-conforming finite elements is quite delicate, especially in the three-dimensional case. Here, we explicitly describe an implementation strategy, which has been embedded in the open source finite element software FreeFem++ (http://www.freefem.org/ff++/). In particular, we use the inverse of a generalized Vandermonde matrix to build basis functions in duality with the degrees of freedom, resulting in an easy-to-use but powerful interpolation operator. We carefully address the problem of applying the same Vandermonde matrix to possibly differently oriented tetrahedra of the mesh over the computational domain. We investigate the preconditioning for Maxwell's equations in the time-harmonic regime, which is an underdeveloped issue in the literature, particularly for high order discretizations. In the numerical experiments, we study the effect of varying several parameters on the spectrum of the matrix preconditioned with overlapping Schwarz methods, both for 2d and 3d waveguide configurations.

1. Introduction

Developing high-speed microwave field measurement systems for wireless, medical or engineering industries is a challenging task. These systems often rely on high frequency (from 1 to 60 GHz) electromagnetic wave propagation in waveguides, and the underlying mathematical model is given by Maxwell's equations. High order finite element (FE) methods make it possible, for a given precision, to reduce significantly the number of unknowns, and they are particularly well suited to discretize wave propagation problems since they can provide a solution with very low dispersion and dissipation errors. However, the resulting algebraic linear systems can be ill conditioned, so that preconditioning becomes mandatory when using iterative solvers.

An appropriate choice to describe the electric field solution of a waveguide propagation problem is a discretization by edge Whitney finite elements [1,2]. Here we consider the high order version of these FEs developed in [3,4] (for other possible high order FE bases see for example [5–9]). We added high order edge FEs to the open source software FreeFem++ [10].

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FreeFem++ is a domain specific language (DSL) specialized in solving boundary value problems by using variational methods, and it is based on a natural transcription of the weak formulation of the considered boundary value problem. The user can add new finite elements to FreeFem++ by defining two main ingredients: the basis functions and an interpolation operator. The basis functions in FreeFem++ are constructed locally, i.e. in each simplex (triangle or tetrahedron), without the need of a transformation from the reference simplex; the chosen definition of high order basis functions fits perfectly this local construction feature since it involves only the barycentric coordinates of the simplex. For the definition of the interpolation operator, in the high order case we need a generalized Vandermonde matrix, introduced for example in [11], to have basis functions in duality with the chosen degrees of freedom. In the case of barycentric coordinates the generalized Vandermonde matrix is independent of the simplex, up to a renumbering of its vertices that we carefully address here. A construction of high order edge finite elements using Cartesian coordinates can be found in [12].

For Maxwell's equations in the *time domain*, for which an implicit time discretization yields at each step a positive definite problem, there are many good solvers and preconditioners in the literature: multigrid or auxiliary space methods, see e.g. [13] for low order finite elements and [14] for high order ones, and Schwarz domain decomposition methods, see e.g. [15]. In this paper, we are interested in solving Maxwell's equations in the *frequency domain*, also called the *time-harmonic* Maxwell's equations: these involve the inherent difficulties of the *indefinite* Helmholtz equation, which is difficult to solve for high frequencies with classical iterative methods [16]. It is widely recognized that domain decomposition methods or preconditioners are key in solving efficiently Maxwell's equations in the time-harmonic regime. The first domain decomposition method for the time-harmonic Maxwell's equations was proposed by Després in [17]. Over the last decade, optimized Schwarz methods were developed, see for example [18–20] and the references therein.

The development of Schwarz algorithms and preconditioners for *high order* discretizations is still an open issue. A recent work for the non overlapping case is reported in [21]. In the present work, we use overlapping Schwarz preconditioners based on impedance transmission conditions for high order discretizations of the curl–curl formulation of time-harmonic Maxwell's equations. Note that domain decomposition preconditioners are suited by construction to parallel computing, which is necessary for large scale simulations. The coupling of high order edge finite elements with domain decomposition preconditioners studied in this paper has been applied in [22] to a large scale problem, coming from a practical application in microwave brain imaging: there, it is shown that the high order approximation of degree 2 makes it possible to attain a given accuracy with much fewer unknowns and much less computing time than the lowest order approximation.

The paper is organized as follows. In Section 2 we introduce the waveguide time-harmonic problem and its variational formulation. In Section 3 we recall the definition of basis functions and degrees of freedom that we adopted here as high order edge FEs. Then, in Section 4 we describe in detail the implementation issues of these FEs, the strategy developed to overcome those difficulties and the ingredients to add them as a new FE in FreeFem++. The overlapping Schwarz preconditioners we used are described in Section 5, followed in Section 6 by the numerical experiments, both in two and three dimensions.

2. The waveguide problem

Waveguides are used to transfer electromagnetic power efficiently from one point in space, where an antenna is located, to another, where electronic components treat the in/out information. Rectangular waveguides, which are considered here, are often used to transfer large amounts of microwave power at frequencies greater than 2 GHz. In this section, we describe in detail the derivation of the simple but physically meaningful boundary value problem which simulates the electromagnetic wave propagation in such waveguide structures. To work in the frequency domain, we restrict the analysis to a time-harmonic electromagnetic field varying with an angular frequency $\omega > 0$. For all times $t \in \mathbb{R}$, we consider the representation of the electric field $\boldsymbol{\mathcal{E}}$ and the magnetic field $\boldsymbol{\mathcal{H}}$ as $\boldsymbol{\mathcal{E}}(\mathbf{x}, t) = \Re(\mathbf{E}(\mathbf{x})e^{\mathrm{i}\omega t})$, $\boldsymbol{\mathcal{H}}(\mathbf{x}, t) = \Re(\mathbf{H}(\mathbf{x})e^{\mathrm{i}\omega t})$, where $\mathbf{E}(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ are the complex amplitudes, for all $\mathbf{x} \in \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^3$ being the considered physical domain. The mathematical model is thus given by the (first order) *time-harmonic Maxwell's equations*:

$$\nabla \times \mathbf{H} = \mathrm{i}\omega\varepsilon_\sigma \mathbf{E}, \qquad \nabla \times \mathbf{E} = -\mathrm{i}\omega\mu\mathbf{H},$$

where μ is the magnetic permeability and ε_σ the electric permittivity of the considered medium in $\mathcal{D}$. To include dissipative effects, we work with a complex valued ε_σ, related to the dissipation-free electric permittivity ε and the electrical conductivity σ by the relation $\varepsilon_\sigma = \varepsilon - \mathrm{i}\frac{\sigma}{\omega}$. This assumption holds in the regions of $\mathcal{D}$ where the current density $\mathbf{J}$ is of conductive type, that is, $\mathbf{J}$ and $\mathbf{E}$ are related by Ohm's law $\mathbf{J} = \sigma\mathbf{E}$. Both ε and μ are assumed to be positive, bounded functions. Expressing Maxwell's equations in terms of the electric field, and supposing that μ is constant, we obtain the *second order (or curl–curl) formulation*

$$\nabla \times (\nabla \times \mathbf{E}) - \gamma^2\mathbf{E} = \mathbf{0}, \tag{1}$$

where the (complex-valued) coefficient $\gamma = \omega\sqrt{\mu\varepsilon_\sigma}$, with $\varepsilon_\sigma = \varepsilon - \mathrm{i}\,\sigma/\omega$. Note that if $\sigma = 0$, we have $\gamma = \tilde{\omega}$, $\tilde{\omega} = \omega\sqrt{\mu\varepsilon}$ being the wavenumber.

Eq. (1) is to be solved in a suitable bounded section Ω of the physical domain $\mathcal{D}$, as shown in Fig. 1. In the 3d case, the physical domain $\mathcal{D} \subset \mathbb{R}^3$ is an infinite 'parallelepiped' parallel to the x-direction and the computational domain is a bounded section, say $\Omega = (0, \mathtt{X}) \times (0, \mathtt{Y}) \times (0, \mathtt{Z}) = (0, \mathtt{c}) \times (0, \mathtt{b}) \times (0, \mathtt{a})$ of $\mathcal{D}$. In the 2d case, the physical domain $\mathcal{D} \subset \mathbb{R}^3$ is the space contained between two infinite parallel metallic plates, say $y = 0$, $y = \mathtt{b}$, and all physical parameters μ, σ, ε have