Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

PDE formulation of some SABR/LIBOR market models and its numerical solution with a sparse grid combination technique☆

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ARTICLE INFO

Article history:
Received 7 November 2016
Received in revised form 8 November 2017
Accepted 20 November 2017
Available online xxxx

Keywords:
Stochastic volatility models
SABR/LIBOR market models
High dimensional PDEs
Sparse grids
Combination technique

ABSTRACT

SABR models have been used to incorporate stochastic volatility to LIBOR market models (LMM) in order to describe interest rate dynamics and price interest rate derivatives. From the numerical point of view, the pricing of derivatives with SABR/LIBOR market models (SABR/LMMs) is mainly carried out with Monte Carlo simulation. However, this approach could involve excessively long computational times. For first time in the literature, in the present paper we propose an alternative pricing based on partial differential equations (PDEs). Thus, we pose original PDE formulations associated to the SABR/LMMs proposed by Hagan and Lesniewsk (2008), Mercurio and Morini (2009) and Rebonato and White (2008). Moreover, as the PDEs associated to these SABR/LMMs are high dimensional in space, traditional full grid methods (like standard finite differences or finite elements) are not able to price derivatives over more than three or four underlying interest rates. In order to overcome this curse of dimensionality, a sparse grid combination technique is proposed. A comparison between Monte Carlo simulation results and the ones obtained with the sparse grid technique illustrates the performance of the method.

1. Introduction

The LMM [1–3] has become the most popular interest rate model. The main reason is the agreement between this model and Black's formulas [4]. The standard LIBOR market model considers constant volatilities for the forward rates. However, this is a very limited hypothesis since it is impossible to reproduce market volatility smiles.

Among the different stochastic volatility models offered in the literature, the SABR model proposed by Hagan, Kumar, Lesniewski and Woodward [5] in the year 2002 stands out for becoming the market standard to reproduce the price of European options. SABR is the acronym for Stochastic, Alpha, Beta and Rho, three of the four model parameters. The SABR model cannot be used to price derivatives whose payoff depends on several forward rates. In fact, SABR model works in the terminal measure, under which both the forward rate and its volatility are martingales. This can always be done if we work with one forward rate in isolation at a time. Under this same measure, however, the process for another forward rate and for its volatility would not be driftless.

In order to allow LMM to fit market volatility smiles, different extensions of the LMM that incorporate the volatility smile by means of the SABR model were proposed. These models are known as SABR/LIBOR market models (SABR/LMMs). In this article we will deal with the models proposed by Hagan [6], Mercurio and Morini [7] and Rebonato [8].

☆ Partially financed by Spanish Grant MTM2013-47800-C2-1-P and by Xunta de Galicia (Grant CN2014/044). First author has also been funded by a FPU Spanish Grant (AP2012-4975).

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https://doi.org/10.1016/j.camwa.2017.11.024
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While Monte Carlo [9] simulation remains the common choice for pricing interest rate derivatives within SABR/LMM setting, several difficulties motivate to address alternative approaches based on PDE formulations. The first issue is that the convergence of Monte Carlo methods, although it depends only very weakly on the dimension of the problem, is very slow. Indeed, if the standard deviation of the result using a single simulation is ϵ then the standard deviation of the error after N simulations is $\epsilon/\sqrt{N}$. Therefore, to improve the accuracy of the solution by a factor of 10, 100 times as many simulations must be performed. The second drawback of Monte Carlo methods is the valuation of options with early-exercise, like in the case of the American options, due to the so-called "Monte Carlo on Monte Carlo" effect. Available Monte Carlo methods for American options are also quite costly, see [10] for example. In contrast, the modification of the PDE to a linear complementarity problem is usually straightforward. Finally, the weakest point of Monte Carlo methods appears to be the computation of the sensitivities of the solution with respect to the underlyings, the so-called "Greeks", which are very used by traders, and are directly given by the partial derivatives of the PDE solution. Besides, path-dependent options, like barrier options, can be easily priced in the PDE context where only the boundary conditions need to be changed, in contrast to Monte Carlo methods, where Brownian bridge techniques [11] must be applied.

In view of previous arguments, in the present paper we pose equivalent PDE formulations for the three above mentioned SABR/LMMs. As far as we now, this is the first time in the literature that these PDE formulations are posed. From the numerical point of view, one main difficulty in these PDE formulations lies in their high dimensionality in space-like variables. In order to cope with this so-called *curse of dimensionality* several methods are available in the literature, see [12,13] for example, which can be put into three categories. The first group uses the Karhunen–Loeve transformation to reduce the stochastic differential equation to a lower dimensional equation, therefore this results in a lower dimensional PDE associated to the previously reduced SDE. The second category gathers those methods which try to reduce the dimension of the PDE itself, like for example dimension-wise decomposition algorithms. Finally, the third category groups the methods which reduce the complexity of the problem in the discretization layer, like for example the method of sparse grids, which we use in the present article.

The sparse grid method was originally developed by Smolyak [14], who used it for numerical integration. It is mainly based on a hierarchical basis [15,16], a representation of a discrete function space which is equivalent to the conventional nodal basis, and a sparse tensor product construction. Zenger [17] and Bungartz and Griebel [18] extended this idea and applied sparse grids to solve PDEs with finite elements, finite volumes and finite differences methods. Besides working directly in the hierarchical basis, the sparse grid can also be computed using the combination technique [19] by linearly combining solutions on traditional Cartesian grids with different mesh widths. This is the approach we follow in this article. Recently, this technique has been used for a financial application related to the pricing of basket options in [20–22].

The paper is organized as follows. In Section 2 some basic concepts related to interest rate derivatives and the corresponding terminology and notation are introduced. In Section 3 we pose the PDE formulations for the SABR/LMMs. In Section 4 we describe the use of a full grid finite differences scheme for the Mercurio and Morini model, the application of which is analogous for the other two SABR/LMMs. Numerical results show the limitations of the full grid method when the number of forward rates increases. Therefore, in Section 5 we describe the sparse grid combination technique applied to the SABR/LMM and show numerical results that illustrate the behavior of the method when the number of forward rates increases. For this purpose, a comparison with Monte Carlo simulation results is used when analytic expressions of the solution are not available, as it happens in most of the cases. Note that Monte Carlo techniques are the usual alternative to price with SABR/LMM.

2. Interest rate derivatives. Caplets and swaptions

This section provides a brief introduction to the interest rate derivatives we deal with in the present article, for a deeper study we refer the reader to [4]. Interest rate derivatives consist of financial contracts that depend on some interest rates.

A zero coupon *bond* with maturity at time T is a contract that pays its holder one unit of currency at time T. The value of this product at time $t < T$ is denoted by $P(t,T)$, and is called the discount factor from time T to time t. Note that $P(T,T) = 1$ for all T.

A *tenor structure* is a set of ordered payment dates $\{T_i,\ i = 0,\dots,N\}$, such that

$$T_0 < T_1 < \cdots < T_{N-1} < T_N.$$

The time between the payment dates is denoted by $\tau_i = T_{i+1} - T_i$. In terms of the corresponding discount factor, a payment of x units at time T_i is worth $xP(t,T_i)$ at time $t < T_i$.

A *forward* interest rate $F_i(t)$ is an interest rate we can contract in order to borrow or lend money during the future time period $[T_i, T_{i+1}]$, and can be expressed in terms of discount factors in the form:

$$F_i(t) = F(t; T_i, T_{i+1}) = \frac{1}{\tau_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right) \text{ where } t \le T_i.$$

Conversely, the price of a bond at time T_i that matures at T_j, $P(T_i,T_j)$, can be expressed in terms of forward LIBOR rates as follows:

$$P(T_i,T_j) = \prod_{k=i}^{j-1} \frac{1}{1+\tau_k F_k(T_i)}.$$

Please cite this article in press as: J.G. López-Salas, C. Vázquez, PDE formulation of some SABR/LIBOR market models and its numerical solution with a sparse grid combination technique, Computers and Mathematics with Applications (2017), https://doi.org/10.1016/j.camwa.2017.11.024.