ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Localized radial basis functions-based pseudo-spectral method (LRBF-PSM) for nonlocal diffusion problems

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ARTICLE INFO

Article history:
Received 25 May 2017
Received in revised form 4 October 2017
Accepted 26 November 2017
Available online xxxx

Keywords:
Spectral/pseudo-spectral methods
Radial basis functions
Cardinal function
Nonlocal diffusion equations
Discontinuity

ABSTRACT

Spectral/pseudo-spectral methods based on high order polynomials have been successfully used for solving partial differential and integral equations. In this paper, we will present the use of a localized radial basis functions-based pseudo-spectral method (LRBF-PSM) for solving 2D nonlocal problems with radial nonlocal kernels. The basic idea of the LRBF-PSM is to construct a set of orthogonal functions by RBFs on each overlapping sub-domain from which the global solution can be obtained by extending the approximation on each sub-domain to the entire domain. Numerical implementation indicates that the proposed LRBF-PSM is simple to use, efficient and robust to solve various nonlocal problems.

1. Introduction

Classical diffusion models are described by partial differential equations. However, these models cannot provide a proper description of problems involving discontinuities, such as diffusion through heterogeneous materials. Some nonlocal diffusion models based on integral operators and fractional derivatives have been proposed for these cases. These models have been used in many fields, including diffusion processes [1–6] and image processing [7,8].

For the numerical solution of nonlocal diffusion models, some mesh-based, including Galerkin finite element methods, discontinuous Galerkin finite element methods, and finite element collocation methods have been developed and analysed [9–16]. The main deficiency of these mesh-based methods is that these numerical methods are dependent on meshes or elements. The generation of an appropriate mesh or element is rather difficult task in three dimensions, and virtually impossible for higher-dimensional problems. If the geometry of the problem is complex, mesh generation, modification and re-meshing become difficult and time-consuming. Therefore, the mesh or element requirement limits the extended use of mesh based methods to models in higher dimensions or with complex geometries.

The recent rapid development of various meshless methods has attracted increasing attentions from researchers in the hope to overcome the mentioned shortcomings. Meshless methods have achieved remarkable progress in recent years. Main efforts have been focusing on different approximation methods over a cluster of scattered nodes. For example, diffuse element method (DEM) [17] was the first meshless method to employ a moving least-squares approximation (MLS) in constructing the shape functions over scattered points instead of an element. Smooth particle hydrodynamics (SPH)

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https://doi.org/10.1016/j.camwa.2017.11.030
0898-1221/

Please cite this article in press as: W. Zhao, et al., Localized radial basis functions-based pseudo-spectral method (LRBF-PSM) for nonlocal diffusion problems, Computers and Mathematics with Applications (2017), https://doi.org/10.1016/j.camwa.2017.11.030.

method [18,19] is one of the well-developed meshless methods. It is based on a kernel approximation. Reproducing kernel particle method (RKPM) [20] further extended the SPH method with improved continuity and multiple scales. These methods are not truly meshless since they use a background mesh for the numerical integration. Radial basis functions are also attractive in the development of meshless methods. The methods based on the radial basis functions are truly meshless since only a set of distinct centres is needed to construct finite dimensional spaces. There are two popular ways based on radial basis functions. One is the Galerkin-type method. There are very few papers on the use of RBFs in the context of Galerkin approximation [21] and combination of RBFs with Galerkin methods to solve partial differential equations was first presented in [22]. Some above meshless methods, such as SPH method, RKPM and RBF Galerkin method, have been used for solving nonlocal problems [23–26]. In this paper, we mainly focus on the other method, the collocation method. The radial basis function (RBF) collocation method is considered as a powerful tool for the approximation of scattered data and for the solution of various kinds of scientific problem [22,27–29]. It is an extremely flexible interpolation method because it does not depend on the locations of the approximation points and applies immediately for scattered data. Application and convergence do not differ much in case of uniform and non-uniform discretizations [30]. In addition, this method generalizes in any number of dimensions and is analogous to the one-dimensional case. Thus, the RBF collocation method is effective for models in multi-dimensions. Comparison of the RBF Galerkin method, the collocation method is cost-effective as no integrations are required.

The pseudo-spectral methods, also known as spectral collocation methods, are known as highly accurate solvers for partial differential equations. This method uses Chebyshev polynomial approximation and generates approximations for the higher order derivatives through successive differentiation of the approximate solution. However, the use of polynomials limits the algorithm to univariate systems constraining it to tensor product grids for multi-dimensions. In some recent cases where both RBF and pseudo-spectral methods have applied, the RBF-based pseudo-spectral method (RBF-PSM) with the help of radial basis functions is to construct shape functions which satisfies the cardinal condition and RBF was found to offer higher accuracy for the same number of points [31,32]. Although this method has shown to be efficient and effective to represent the approximate solution of PDEs, the development of mathematical theory on stability and convergence order estimates, however, is still far from satisfactory. This approach always generates dense or full matrices which require $\mathcal{O}(N^2)$ of memory where N is the number of unknowns in the discretized system. Many fast algorithms have been developed for dense structured matrices [33], including the conjugate gradient(CG) method [34], bi-conjugate gradient(BCG) method [35] and generalized minimal residual(GMRES) method [36]. However, if the equation contains a complex operator, such as integral-type operator, the generation of the linear system still consumes a lot of time. This also limits the applicability of the RBF-PS method to solve large-scale problems.

In this paper, we present a localized RBF-PSM (LRBF-PSM) to reduce the computational cost. This method is also called local radial basis functions collocation method (LRBFCM) in [37,38]. The basic idea of this approach is to apply collocation separately on each overlapping sub-domain of the whole domain. This localized method keeps the meshlessness and high accuracy of the RBF method and has been applied successfully to solve large scale physical problems [29,39,40]. The purpose of this paper is to apply the LRBF-PSM for solving nonlocal problems.

The rest of the paper is organized as follows: Section 2 gives a brief review of nonlocal diffusion models. In Section 3, we describe the RBF-PSM at first and present the LRBF-PSM. The corresponding discrete forms for nonlocal models are given in Section 4. The numerical results are then reported and commented in Section 5 while the conclusions of the present work and possible future developments are given in the final Section 6.

2. Mathematical model: nonlocal diffusion equation

Let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, compactly supported, radial, continuous function with $\int_{\mathbb{R}^d} \gamma(\mathbf{z})d\mathbf{z} = 1$. A nonlocal diffusion equation is given by

$$u_t(\mathbf{x}, t) = (\gamma * u - u)(\mathbf{x}, t) = \int_{\mathbb{R}^d} \gamma(\mathbf{x} - \mathbf{x}')u(\mathbf{x}', t)d\mathbf{x}' - u(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T], \tag{2.1}$$

where $T > 0$ and $u_t(\mathbf{x}, t) = \partial u(\mathbf{x}, t)/\partial t$. According to [5,41,42], if $u(\mathbf{x}, t)$ is a density at $(\mathbf{x}, t)$ and $\gamma(\mathbf{x} - \mathbf{x}')$ is the probability distribution of jumping from $\mathbf{x}'$ to $\mathbf{x}$, then $(\gamma * u)(\mathbf{x}, t) = \int_{\mathbb{R}^d} \gamma(\mathbf{x} - \mathbf{x}')u(\mathbf{x}', t)d\mathbf{x}'$ is the rate of individuals arriving at $\mathbf{x}$ from all other places and $u(\mathbf{x}, t) = \int_{\mathbb{R}^d} \gamma(\mathbf{x} - \mathbf{x}')u(\mathbf{x}, t)d\mathbf{x}'$ is the rate of individuals leaving $\mathbf{x}$ to all other places. Eq. (2.1) is called nonlocal diffusion equation since the diffusion density u at point $\mathbf{x}$ at time t does not only depend on $u(\mathbf{x}, t)$ but on all values of u in a neighbourhood of $\mathbf{x}$ through the convolution $\gamma * u$. Let $\Omega \subset \mathbb{R}^d$ denote a bounded, open domain and δ is the compactly supported radius of γ, then (2.1) is expressed as

$$u_t(\mathbf{x}, t) = \int_{B_\delta(\mathbf{x})} (u(\mathbf{x}', t) - u(\mathbf{x}, t))\gamma(\mathbf{x} - \mathbf{x}')d\mathbf{x}', \quad \mathbf{x} \in \Omega, t > 0, \tag{2.2}$$

where $B_\delta(\mathbf{x})$ represents the neighbourhood of $\mathbf{x}$, which is often chosen to be an open neighbourhood that is centred at $\mathbf{x}$ with radius δ. For 2D case, the neighbourhood is a disc (see Fig. 1). The δ is also called the horizon size of the material. If $u_t(\mathbf{x}, t)$ is replaced by $u_{tt}(\mathbf{x}, t)$, a vector valued form of (2.2) can be equivalent to the peridynamics model for the mechanics of linear materials [43]. Furthermore, we refer to $\Omega_{\mathcal{I}}$ as the interaction domain, which is the nonlocal analogue of the boundary and is necessary to impose the nonlocal analogue of boundary conditions.

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