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Superconvergence in a DPG method for an ultra-weak formulation

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ABSTRACT

In this work we study a DPG method for an ultra-weak variational formulation of a reaction–diffusion problem. We improve existing a priori convergence results by sharpening an approximation result for the numerical flux. By duality arguments we show that higher convergence rates for the scalar field variable are obtained if the polynomial order of the corresponding approximation space is increased by one. Furthermore, we introduce a simple elementwise postprocessing of the solution and prove superconvergence. Numerical experiments indicate that the obtained results are valid beyond the underlying model problem.

1. Introduction

The discontinuous Petrov Galerkin method with optimal test functions (DPG method), introduced in the present form by Demkowicz & Gopalakrishnan in a series of papers [1,2] and [3,4], has gained much attention due to its inherent stability properties. This makes the method particularly interesting for singularly perturbed problems [5–7], where the aim is to provide robust error bounds. Other features of the DPG methods are, for instance, its equivalence to a minimal residual method or its "built-in" error estimator, for the latter see [8].

DPG methods use broken test spaces, which allow for an efficient calculation of (nearly) optimal test functions and the feasible evaluation of the residual in the dual norm of the test space. Different variational formulations can be used within the DPG framework. Very popular is the ultra-weak formulation, see [9] for the analysis in the case of a Poisson model problem. For second order elliptic problems, ultra-weak formulations are derived by recasting the PDE into a first-order system, multiplying with test functions, and then integrating by parts elementwise. This requires the introduction of new trace unknowns that only live on the skeleton of the mesh. These variables represent the trace resp. normal flux of the solution on element boundaries. Another DPG method based on a variational formulation without recasting the PDE into a first-order system is introduced and analyzed for the Poisson model problem in [10].

The purpose of this note is to study convergence rates for DPG methods using the ultra-weak formulation. One particular drawback of ultra-weak formulations is that u and its gradient $\boldsymbol{\sigma} = \nabla u$ are usually approximated simultaneously with the same order. Therefore, we are interested if higher convergence rates for the scalar field variable u are possible, either by increasing the polynomial order of the corresponding approximation space or by defining an approximation of the scalar field variable by a suitable postprocessing. The latter issue has been raised and addressed for other numerical schemes too, e.g., the HDG method [11]. In the context of DG methods, the achievement of higher convergence rates by postprocessing the solution has been studied thoroughly and is called superconvergence, see, e.g. [12–14].

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Let us also note that convergence rates for primal DPG methods were studied and analyzed in [15]. There, duality arguments have been introduced and used to prove higher convergence rates of the field variable in weaker norms. We point out that by using an ultra-weak formulation, we only have weak norms at hand and thus the ideas from [15] cannot directly be transferred to our situation. In particular, higher convergence rates for ultra-weak formulations can only be obtained if we augment the approximation space of the scalar field variable or by a postprocessing of the solution.

1.1. Overview and main contributions of this work

The remainder of this note and the main results are given as follows:

- In Section 2 we introduce the model problem together with the DPG method based on the ultra-weak variational formulation. We also recall some important results from the literature that are used in the proofs that follow.
- Section 3 deals with a priori convergence results for the overall error. We improve the existing analysis in the sense that only minimal regularity of the solution u is required to obtain optimal convergence rates.
- In Section 4 we augment the trial space for approximations $u_h \approx u$, i.e., we seek u_h in a polynomial space of order $p+1$ and $\boldsymbol{\sigma}_h \approx \boldsymbol{\sigma} = \nabla u$ in a polynomial space of order p. We develop duality arguments and prove in Theorem 10 that the error $u - u_h$ in the $L^2(\Omega)$ norm converges at a higher rate than the overall error (which also includes the error for the skeleton unknowns). The practical relevance of augmenting the trial space is that existing finite element codes for DPG methods can be used with minor modification.
- In Section 5 we seek approximations $u_h \approx u$ and $\boldsymbol{\sigma}_h \approx \boldsymbol{\sigma}$ in polynomial approximation spaces of the same order p. Then, we introduce the approximation $\widetilde{u}_h$ by postprocessing the solution components $u_h, \boldsymbol{\sigma}_h$. To be more precise, $\widetilde{u}_h$ is elementwise given as the solution of a discrete Neumann problem. In Theorem 13 we prove superconvergence, i.e., the error $u - \widetilde{u}_h$ in the $L^2(\Omega)$ norm converges at a higher rate than the overall error.
- Finally, numerical experiments are presented in Section 6. We consider one problem in a convex domain and one problem in a nonconvex domain. For the latter, reduced convergence rates are predicted by our analysis and also observed in the example.

1.2. Notational convention

In the remainder of this work we will write $A \lesssim B$ resp. $B \lesssim A$ if there exists a constant $C > 0$ that is independent of the maximal mesh-size h such that $A \le CB$ resp. $B \le CA$. Moreover, $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. This notation will be heavily used in the proofs, whereas in all statements of the following theorems and corollaries we explicitly point out the dependence of $C > 0$ on relevant quantities.

2. DPG with ultra-weak formulation

2.1. Model problem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded simply connected Lipschitz domain with polygonal boundary $\Gamma = \partial\Omega$. We consider the reaction–diffusion problem

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{1a}$$
$$u = 0 \quad \text{on } \Gamma. \tag{1b}$$

It is well known that this problem admits for all $f \in L^2(\Omega)$ a unique solution $u \in H_0^1(\Omega)$. Moreover, it is known that for some $s \in (1/2, s_\Omega]$ it holds that

$$u \in H^{1+s}(\Omega) \quad \text{and} \quad \|u\|_{H^{1+s}(\Omega)} \le C\|f\|,$$

where $s_\Omega \in (1/2, 1]$ depends on the geometry of Ω. For a convex domain Ω it even holds that

$$u \in H^2(\Omega) \quad \text{and} \quad \|u\|_{H^2(\Omega)} \le C\|f\|.$$

Here, and throughout $H^t(\Omega)$ denotes the Sobolev space of real order $t > 0$ with norm $\|\cdot\|_{H^t(\Omega)}$ and seminorm $|\cdot|_{H^t(\Omega)}$.

2.2. Notation

With $\|\cdot\|$ resp. $(\cdot\,,\cdot)$ we denote the canonical norm resp. scalar product on $L^2(\Omega)$. Let $\mathcal{T}$ denote a mesh on Ω consisting of triangles, i.e., $\bigcup_{T\in\mathcal{T}} \overline{T} = \overline{\Omega}$. Throughout, we assume that the mesh $\mathcal{T}$ is shape-regular and set $h := \max_{T\in\mathcal{T}} \mathrm{diam}(T)$. With $\|\cdot\|_T$ resp. $(\cdot\,,\cdot)_T$ we denote the norm resp. scalar product on $L^2(T)$ for $T \in \mathcal{T}$.

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