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Convergence of an implicit–explicit midpoint scheme for computational micromagnetics

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Based on lowest-order finite elements in space, we consider the numerical integration of the Landau–Lifschitz–Gilbert equation (LLG). The dynamics of LLG is driven by the so-called effective field which usually consists of the exchange field, the external field, and lower-order contributions such as the stray field. The latter requires the solution of an additional partial differential equation in full space. Following Bartels and Prohl (2006), we employ the implicit midpoint rule to treat the exchange field. However, in order to treat the lower-order terms effectively, we combine the midpoint rule with an explicit Adams–Bashforth scheme. The resulting integrator is formally of second-order in time, and we prove unconditional convergence towards a weak solution of LLG. Numerical experiments underpin the theoretical findings.

1. Introduction

Time-dependent micromagnetic phenomena are usually modeled by the Landau–Lifschitz–Gilbert equation (LLG); see (1). This nonlinear partial differential equation (PDE) describes the behavior of the magnetization of some ferromagnetic body under the influence of the so-called effective field $\boldsymbol{h}_{\mathrm{eff}}$. Global-in-time existence (and possible nonuniqueness) of weak solutions of LLG goes back to [1,2]. For smooth problems, LLG admits a unique strong solution locally in time, provided the initial data are smooth (cf. [3]). Under similar restrictions the recent work [4] proves a strong–weak uniqueness principle for LLG. Unconditionally convergent numerical integrators have first been analyzed mathematically in [5,6], where $\boldsymbol{h}_{\mathrm{eff}}$ only consists of the exchange field (see Section 2.1). Here, *unconditional convergence* means that convergence of the numerical integrator enforces no CFL-type coupling of the spatial mesh-size h and the time-step size k. Moreover, convergence is understood in the sense that the sequence of discrete solutions for $h, k \to 0$ admits a subsequence which converges weakly in $\boldsymbol{H}^1$ towards a weak solution of LLG. The tangent plane integrator of [6] requires to solve one *linear* system per time-step (posed in the time-dependent discrete tangent plane), but is formally only first-order in time. Instead, the midpoint scheme of [5] is formally second-order in time, but involves the solution of one nonlinear system per time-step.

Usually, the effective field $\boldsymbol{h}_{\mathrm{eff}}$ which drives the dynamics of LLG couples LLG to other stationary or time-dependent PDEs; see, e.g., [7] for the coupling of LLG with the full Maxwell system, [8] for the electron spin diffusion in ferromagnetic multilayers, or [9] for LLG with magnetostriction. In the case that the effective field involves stationary PDEs only (e.g., $\boldsymbol{h}_{\mathrm{eff}}$ consists of exchange field, anisotropy field, applied exterior field, and self-induced stray field), the numerical analysis of the tangent plane integrator of [6] has been generalized in [10,11], where the lower-order contributions are treated explicitly in time by means of a forward Euler step. It is proved that this preserves unconditional convergence. In [12] and [13,14], the

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tangent plane integrator is adapted to the coupling of LLG with the full Maxwell system resp. the eddy current formulation. The works [15,16] extend the tangent plane integrator to LLG with magnetostriction resp. LLG with spin diffusion interaction. Throughout, [12–16] prove unconditional convergence of the overall integrator. Moreover, one general theme of [12,14–16] is that the time marching scheme decouples the integration of LLG and the coupled PDE, so that – despite the possibly nonlinear coupling [15,16] – only two linear systems have to be solved per time-step. Moreover, [16] proves that the nodal projection step of the original tangent plane integrator [6] can be omitted without losing unconditional convergence. For this projection-free variant of the tangent plane integrator, the recent work [17] also proves strong $\boldsymbol{H}^1$-convergence towards strong solutions.

As far as the midpoint scheme from [5] is concerned, the work [18] provides an extended scheme for the Maxwell–LLG system. Even though the decoupling of the nonlinear LLG equation and the linear Maxwell system appears to be of interest for a time-marching scheme, the analysis of [18] treats only the full nonlinear system in each time-step.

The present work transfers ideas and results from [10,11] for the tangent plane integrator to the midpoint scheme. We prove that lower-order terms can be treated explicitly in time. This dramatically lowers the computational work to solve the nonlinear system in each time-step of the midpoint scheme. Unlike [10,11], however, the effective treatment of the lower-order terms requires an explicit two-step method (instead of the simple forward Euler method) to preserve the second-order convergence of the midpoint scheme. We prove that such an approach based on the Adams–Bashforth scheme guarantees unconditional convergence and remains formally of second-order in time. As an application of the proposed general framework, we discuss the discretization of the extended form of LLG [19,20] which is used to describe the current driven motion of domain walls.

2. Model problem and discretization

This section states the Gilbert formulation of LLG and extends the notion of a weak solution from [2] to the present situation. Then, we introduce the notation for our finite element discretization and formulate the numerical integrator. Throughout, we employ standard Lebesgue and Sobolev spaces $L^2(\Omega)$ resp. $H^1(\Omega)$. For any Banach space B, we let $\boldsymbol{B} := B^3$, e.g., $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

2.1. Model problem

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, initial data $\boldsymbol{m}^0 \in \boldsymbol{H}^1(\Omega)$, final time $T > 0$, and the Gilbert damping constant $\alpha > 0$, the Gilbert form of LLG reads

$$\partial_t \boldsymbol{m} = -\boldsymbol{m} \times \boldsymbol{h}_{\mathrm{eff}} + \alpha\, \boldsymbol{m} \times \partial_t \boldsymbol{m} \qquad \text{in } \Omega_T := (0,T)\times\Omega, \tag{1a}$$

$$\partial_{\mathbf{n}} \boldsymbol{m} = \boldsymbol{0} \qquad \text{on } (0,T)\times\partial\Omega, \tag{1b}$$

$$\boldsymbol{m}(0) = \boldsymbol{m}^0 \qquad \text{in } \Omega. \tag{1c}$$

With $C_{\mathbf{ex}} > 0$, $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$, and $\boldsymbol{\pi} : \boldsymbol{H}^1(\Omega) \cap \boldsymbol{L}^\infty(\Omega) \to \boldsymbol{L}^2(\Omega)$, the effective field reads

$$\boldsymbol{h}_{\mathrm{eff}} := C_{\mathbf{ex}} \Delta \boldsymbol{m} \;+\; \boldsymbol{\pi}(\boldsymbol{m}) + \boldsymbol{f}; \tag{2}$$

see Theorem 4 for further assumptions on $\boldsymbol{\pi}(\cdot)$ and $\boldsymbol{f}$. With the $\boldsymbol{L}^2$-scalar product $\langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle := \int_\Omega \boldsymbol{\varphi}\cdot\boldsymbol{\psi}\,\mathrm{d}x$ for all $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$, consider the bulk energy

$$\mathcal{E}(\boldsymbol{m},\boldsymbol{f}) := \frac{C_{\mathbf{ex}}}{2}\|\nabla \boldsymbol{m}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\langle \boldsymbol{\pi}(\boldsymbol{m}), \boldsymbol{m}\rangle - \langle \boldsymbol{f}, \boldsymbol{m}\rangle. \tag{3}$$

With the convention $\langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla \boldsymbol{\varphi}\rangle := \sum_{\ell=1}^3 \langle \boldsymbol{m} \times \partial_{x_\ell} \boldsymbol{m}, \partial_{x_\ell} \boldsymbol{\varphi}\rangle$, we follow [2] for the definition of a weak solution to (1). Note that the variational formulation (4) is just the weak formulation of (1) after integration by parts.

Definition 1. A function $\boldsymbol{m}$ is a weak solution to (1) if the following properties (i)–(iv) are satisfied:

(i) $\boldsymbol{m} \in \boldsymbol{H}^1(\Omega_T)$ and $|\boldsymbol{m}| = 1$ almost everywhere in Ω_T;
(ii) $\boldsymbol{m}(0) = \boldsymbol{m}^0$ in the sense of traces;
(iii) $\boldsymbol{m}$ has bounded energy in the sense that there exists a constant $C > 0$, which depends only on $\boldsymbol{m}^0$, $\boldsymbol{\pi}(\cdot)$, and $\boldsymbol{f}$, such that, for almost all $\tau \in (0,T)$, it holds that

$$\|\nabla \boldsymbol{m}(\tau)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^\tau \|\partial_t \boldsymbol{m}\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t \le C < \infty;$$

(iv) for all $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega_T)$, it holds that

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{m}, \boldsymbol{\varphi}\rangle \mathrm{d}t &= C_{\mathbf{ex}} \int_0^T \langle \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla\boldsymbol{\varphi}\rangle \mathrm{d}t - \int_0^T \langle \boldsymbol{m} \times \boldsymbol{\pi}(\boldsymbol{m}), \boldsymbol{\varphi}\rangle \mathrm{d}t \\ &\quad - \int_0^T \langle \boldsymbol{m} \times \boldsymbol{f}, \boldsymbol{\varphi}\rangle \mathrm{d}t + \alpha \int_0^T \langle \boldsymbol{m} \times \partial_t \boldsymbol{m}, \boldsymbol{\varphi}\rangle \mathrm{d}t.\end{aligned} \tag{4}$$

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