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A posteriori error estimates of two-grid finite volume element methods for nonlinear elliptic problems☆

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A B S T R A C T

In this paper, we study the a posteriori error estimates of two-grid finite volume element method for second-order nonlinear elliptic equations. We derive the residual-based a posteriori error estimator and prove the computable upper and lower bounds on the error in H^1-norm. The a posteriori error estimator can be used to assess the accuracy of the two-grid finite volume element solutions in practical applications. Numerical examples are provided to illustrate the performance of the proposed estimator.

1. Introduction

In this paper, we will study the a posteriori error estimates of the two-grid finite volume element method for the following two-dimensional nonlinear elliptic boundary value problems

$$\begin{cases} -\nabla \cdot (A(u)\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $x \in \Omega \subset \mathbb{R}^2$ is an open bounded polygonal domain with the boundary $\partial\Omega$ and with $A : \mathbb{R} \to \mathbb{R}$ sufficiently smooth such that there exist constants $\beta_i, i = 1, 2, 3$, satisfying

$$0 < \beta_1 \le A(x) \le \beta_2, \ |A'(x)| \le \beta_3, \ \forall x \in \mathbb{R}. \tag{1.2}$$

Due to the local conservation property and other attractive properties such as robustness with unstructured meshes, finite volume element method is widely used in many fields. Especially for many physical and engineering applications, such as fluid mechanics, heat and mass transfer and petroleum engineering, this numerical conservation property is very crucial. There are a lot of studies of the mathematical analysis for finite volume element method. We can refer to the monograph [1] for the general presentation of this method, to literatures [2–9] and the references therein.

Two-grid method was first introduced by Xu [10,11] to solve the nonsymmetric linear and nonlinear elliptic partial differential equations. The main idea of the two-grid method is to use a coarse grid with grid size H to produce a rough

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approximation of the solution of nonlinear problems firstly. Then we solve a linearized problem with the rough solution as known values on the fine grid with grid size $h < H$ to get a corrected solution. At almost the same time, Huang and Chen [12] have proposed a multilevel iterative method that not only reduces the computing work but also preserves all of the high accuracy properties such as superconvergence, extrapolation, etc. for finite element solutions to singular problems. Later on, two-grid method was further investigated by many authors, for instance, Dawson, Wheeler and Woodward [13,14] for finite difference method and finite element method, Chen and Huang et al. [15–17] for mixed finite element method, Bi, Chen and Zhang et al. [18–21] for finite volume element method.

Adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations since the pioneering work of Babuvška and Rheinboldt [22]. There are many works on the a posteriori error estimates of finite element method [23–26]. The a posteriori error analysis for finite volume element method is also developed in recent years. Lazarov and Tomov [27] have analyzed the a posteriori error estimates and adaptive finite volume element method for the convection–diffusion reaction problems. Afif et al. [28] established residual-type a posteriori error estimates for a linear elliptic boundary value problem. Bergam et al. [29] derived a posteriori error estimates for vertex-centered finite volume method of a class of nonlinear elliptic problems. Carstensen et al. [30] studied residual-type error estimators for a posteriori finite volume element error control with and without upwind scheme for general elliptic problems. Bi and Ginting [31] have constructed residual-type a posteriori error estimate of finite volume element method for a quasi-linear elliptic problem. Ye [32] has established general residual-type a posteriori estimator for the second-order elliptic problem that can be applied to different finite volume methods.

For the nonlinear elliptic problem (1.1), Chatzipantelidis et al. [5] have proved the existence of the finite volume element method and derived error estimates in H^1, L^2 and L^∞ norm. Then Bi and Ginting studied the two-grid finite volume element method for this model and proved the optimal error estimates in the H^1 norm. Based on these results, we study the residual-based a posteriori error estimates of two-grid finite volume element method for the nonlinear elliptic problem. We construct a computational residual-based a posteriori error estimator of the two-grid finite volume element method and develop the global upper and local lower bounds on the error in the H^1-norm. Our theoretical and experiment findings show that the a posteriori error estimates are valid and efficient in the two-grid finite volume element method.

The rest of this paper is organized as follows. In Section 2 we describe the two-grid finite volume element scheme for the nonlinear elliptic problem (1.1) and give some useful lemmas. In Section 3 we propose a residual-based a posteriori error estimator of the two-grid finite volume element method for (1.1) and derive the global upper and local lower bounds on the error in H^1 norm. In Section 4 we give some numerical examples to validate the theoretical results. Finally we summarize the main results of this paper and make a conclusion.

Throughout this paper, we will use C or C with its subscript to denote a generic positive constant. The constant will not depend on the mesh parameters and may represent different values in different places.

2. Two-grid finite volume element method

We shall use the standard notations for Sobolev spaces $W^{s,p}(\Omega)$ [33] and their associated inner products $(\cdot,\cdot)_{s,p,\Omega}$, norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$, respectively. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible; i.e., $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

For the nonlinear elliptic problem (1.1), the weak formulation is to find $u \in H_0^1(\Omega)$ such that

$$a(u; u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product and the bilinear form $a(\cdot;\cdot,\cdot)$ is defined by

$$a(w; u, v) = \int_\Omega A(w)\nabla u \cdot \nabla v \mathrm{d}x, \ \forall u, v, w \in H_0^1(\Omega).$$

The existence and uniqueness of the weak solution of (2.1) has been proved in [34].

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in \mathcal{T}_h$. Based on this triangulation, let V_h be the standard conforming finite element space of piecewise linear functions,

$$V_h = \{v \in C(\Omega) : \ v|_K \text{ is linear}, \forall K \in \mathcal{T}_h; v|_{\partial\Omega} = 0\}.$$

Then the standard finite element method for (2.1) is to find $u_h \in V_h$, such that

$$a(u_h; u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{2.2}$$

where

$$a(w_h; u_h, v_h) = \int_\Omega A(w_h)\nabla u_h \cdot \nabla v_h \mathrm{d}x, \ \forall w_h, u_h, v_h \in V_h.$$

The existence and uniqueness of (2.2) was stated in [11].

For the finite volume element method, we introduce the dual partition $\mathcal{T}_h^*$ of the original partition $\mathcal{T}_h$ which is called control volumes. Now we present the construction of $\mathcal{T}_h^*$. For a triangulation $K \in \mathcal{T}_h$, we choose the barycenter z_K of K and

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