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A stable Gaussian radial basis function method for solving nonlinear unsteady convection–diffusion–reaction equations

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ABSTRACT

We investigate a novel method for the numerical solution of two-dimensional time-dependent convection–diffusion–reaction equations with nonhomogeneous boundary conditions. We first approximate the equation in space by a stable Gaussian radial basis function (RBF) method and obtain a matrix system of ODEs. The advantage of our method is that, by avoiding Kronecker products, this system can be solved using one of the standard methods for ODEs. For the linear case, we show that the matrix system of ODEs becomes a Sylvester-type equation, and for the nonlinear case we solve it using predictor–corrector schemes such as Adams–Bashforth and implicit–explicit (IMEX) methods. This work is based on the idea proposed in our previous paper (2016), in which we enhanced the expansion approach based on Hermite polynomials for evaluating Gaussian radial basis function interpolants. In the present paper the eigenfunction expansions are rebuilt based on Chebyshev polynomials which are more suitable in numerical computations. The accuracy, robustness and computational efficiency of the method are presented by numerically solving several problems.

1. Introduction

We consider the following nonlinear unsteady convection–diffusion–reaction equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1(x)\dfrac{\partial^2 u}{\partial x^2} + a_2(y)\dfrac{\partial^2 u}{\partial y^2} + \nabla\cdot\left(b_1(x,y), b_2(x,y)\right)u + \left(c_1(x) + c_2(y)\right)u \\ + f(t,x,y,u(x,y,t)), \quad (x,y,t) \in \Omega \times (0,T], \\ u(x,y,0) = \psi(x,y), \qquad (x,y) \in \overline{\Omega}, \end{cases} \tag{1}$$

where $a_1(x)$, $a_2(y) > 0$ are the diffusion coefficients, $(b_1(x,y), b_2(x,y))$ is the convection speed vector, $c_1(x)$, $c_2(y)$ are reaction coefficients, Ω is a bounded domain in $\mathbb{R}^2$, and $\psi(x,y)$ is a smooth function. On the boundary $\partial\Omega$, a Robin boundary condition is used:

$$\alpha u(x,y,t) + \beta\frac{\partial u}{\partial \mathbf{n}}(x,y,t) = h(x,y,t), \qquad (x,y,t) \in \partial\Omega \times (0,T],$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary, h is a smooth function, and α, β are not both equal to zero. If $\alpha \neq 0$, $\beta = 0$, the boundary conditions can be imposed as Dirichlet boundary conditions and if $\alpha = 0$, $\beta \neq 0$, they can

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be imposed as Neumann boundary conditions. Convection–diffusion–reaction equations have a wide range of applications in many fields of science as models for flow problems or heat transfer, chemical reaction processes, convective heat transport problems, nuclear reactions, water pollution problems, simulation of oil extraction from underground reservoirs and semiconductor device simulation are governed by these equations (see [1]). Also for applications in financial problems see [2].

In the special case when the convection speed vector is constant, and when the diffusion field (a_1, a_2) is very small in comparison with the convection field (b_1, b_2), the solution of the equation may suffer from boundary and interior layers [1,3]. Boundary layers represent a rapid transition of the solution near the boundary of the domain, which can cause numerical difficulties because the solution changes rapidly over a very small region of the domain. Also, interior layers, which correspond to rapid transitions in the domain, may cause even more numerical difficulties because one often does not know a priori where the interior layer will be. Due to the presence of boundary layers, conventional numerical methods like finite differences and finite elements may produce poor accuracy or suffer from instability. To overcome these difficulties, a class of so-called stabilized finite element methods has been developed [4–6].

Although there are many papers about the solution of one-dimensional convection–diffusion–reaction equations, as far as we know, there are only few papers available for the numerical solution of two-dimensional nonlinear time-dependent convection–diffusion–reaction equations, such as [2,7–10].

In recent years, lots of researchers have focused on stable approaches for evaluating RBF interpolants, such as the Contour–Padé approach [11] or the RBF-QR method [12–14]. In [15], the authors developed a variant of the RBF-QR method by using an eigenfunction expansion of the Gaussian RBF and established a connection between the RBF-QR algorithm and Mercer's theorem. In [16], McCourt used this Gaussian eigenfunction approach to solve boundary value problems. In [17], the authors enhanced the eigenfunction expansion approach for evaluating Gaussian RBF interpolants by taking advantage of the orthogonality of the eigenfunctions which are based on Hermite polynomials and using their zeros for interpolating functions and collocating boundary value problems in one and two dimensional problems. In this paper the eigenfunction expansions are rebuilt based on Chebyshev polynomials, which are more suitable in numerical computations and Eq. (1) is discretized in space by this new approach.

The remainder of this paper is organized as follows. Section 2 is devoted to some preliminary concepts about the stable method for Gaussian RBF interpolation based on eigenfunction expansions. In Section 3, the solution of two-dimensional nonlinear unsteady convection–diffusion–reaction equations is investigated. In Section 4, some numerical experiments that demonstrate the accuracy, efficiency and stability of the proposed method are included.

2. Preliminary discussion of the method

Before discussing the discretization of Eq. (1), we present a brief summary of the new stable method for Gaussian RBF interpolation. According to Mercer's theorem, every positive definite kernel $K : \Omega \times \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^d$, can be represented in terms of the (positive) eigenvalues $\lambda_n \to 0$ and (normalized) eigenfunctions φ_n of an associated compact integral operator [18], i.e.,

$$K(\mathbf{x}, \mathbf{z}) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(\mathbf{x}) \varphi_n(\mathbf{z}). \tag{2}$$

Now suppose $\mathcal{K}$ is a native Hilbert space of functions on Ω, $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \Omega$ is the set of centers and $\mathcal{K}_X = \text{span}\{K(\cdot, \mathbf{x}_j), \ x_j \in X\}$ is the subspace spanned by the basis $K(\cdot, \mathbf{x}_j), \ 1 \le \mathbf{x}_j \le N$. We can write the interpolant $s_f \in \mathcal{K}_X$ of $f \in \mathcal{K}$ at $\mathbf{x}_1, \ldots, \mathbf{x}_N$ as

$$s_f(x) = \sum_{j=1}^{N} c_j K(\mathbf{x}, \mathbf{x}_j),$$

where the coefficients c_j are determined by the interpolation conditions $s_f(\mathbf{x}_j) = f(\mathbf{x}_j) : \ j = 1, \ldots, N$; i.e., they can be obtained by solving the following $N \times N$ linear system

$$\mathbf{Kc} = \mathbf{f},$$

where $\mathbf{f} = [f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N)]^T$, $\mathbf{c} = [c_1, \ldots, c_N]^T$, and

$$\mathbf{K} = \begin{bmatrix} K(\mathbf{x}_1, \mathbf{x}_1) & \cdots & K(\mathbf{x}_1, \mathbf{x}_N) \\ \vdots & & \vdots \\ K(\mathbf{x}_N, \mathbf{x}_1) & \cdots & K(\mathbf{x}_N, \mathbf{x}_N) \end{bmatrix}.$$

In applications, we have to truncate the series in (2). By choosing exactly N terms of this series and ignoring the truncation error, we can approximate the kernel as

$$K(\mathbf{x}, \mathbf{x}_j) = \sum_{n=1}^{N} \lambda_n \varphi_n(\mathbf{x}) \varphi_n(\mathbf{x}_j),$$

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