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A multigrid optimization algorithm for the numerical solution of quasilinear variational inequalities involving the p-Laplacian ☆

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ABSTRACT

In this paper we propose a multigrid optimization algorithm (MG/OPT) for the numerical solution of a class of quasilinear variational inequalities of the second kind. This approach is enabled by the fact that the solution of the variational inequality is given by the minimizer of a nonsmooth energy functional, involving the p-Laplace operator. We propose a Huber regularization of the functional and a finite element discretization for the problem. Further, we analyze the regularity of the discretized energy functional, and we are able to prove that its Jacobian is slantly differentiable. This regularity property is useful to analyze the convergence of the MG/OPT algorithm. In fact, we demonstrate that the algorithm is globally convergent by using a mean value theorem for semismooth functions. Finally, we apply the MG/OPT algorithm to the numerical simulation of the viscoplastic flow of Bingham, Casson and Herschel–Bulkley fluids in a pipe. Several experiments are carried out to show the efficiency of the proposed algorithm when solving this kind of fluid mechanics problems.

1. Introduction

In this paper, we focus on the development of a multigrid algorithm for the fast finite element solution of a class of quasilinear variational inequalities of the second kind. The main idea is the application of an efficient multigrid approach to this kind of problems which, typically, leads us to the solution of large systems.

Let Ω be an open and bounded set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. We are concerned with the numerical solution of the following class of quasilinear variational inequalities of the second kind: find $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla(v-u))\,dx + g\int_\Omega |\nabla v|\,dx - g\int_\Omega |\nabla u|\,dx \geq \int_\Omega f(v-u)\,dx,\ \forall v \in W_0^{1,p}(\Omega),$$

where $1 < p < \infty$, $g > 0$ and $f \in L^q(\Omega)$. Here, $q = \frac{p}{p-1}$ stands for the conjugate exponent of p. It is known that these variational inequalities correspond to a first order necessary optimality condition for the following class of nonsmooth

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optimization problems.

$$\min_{u \in W_0^{1,p}(\Omega)} J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + g\int_\Omega |\nabla u|\, dx - \int_\Omega fu\, dx. \tag{1}$$

Consequently, we focus on the fast solution of this optimization problem. The existence and uniqueness of solutions for this problem has been analyzed and verified in previous contributions, such as [1].

The variational inequalities under study provide a versatile tool in the study of a class of free boundary problems which arise in the modeling of complex fluids and materials. In fact, diverse problems including the flow of viscoplastic materials, the flow of electro- and magneto-rheological fluids and phenomena in glaciology have been successfully simulated by this kind of models [1–3].

Several approaches have been proposed for the numerical solution of problems like (1). In [4] an Augmented Lagrangian method is implemented for the numerical simulation of the flow of viscoplastic materials. In [1], a preconditioned descent algorithm is proposed and analyzed both in finite and infinite dimension settings. Regarding the use of multigrid algorithms, in [5,6] the author proposes algorithms for variational inequalities of the first and the second kind, based on the extended relaxation methods. In [7], the author proposes a multigrid algorithm for variational inequalities of the second kind using a combination of convex minimization with constrained Newton linearization. However, these contributions focus on variational inequalities involving linear elliptic operators such as the Laplacian.

The multigrid approach is a very appealing way to develop fast solution algorithms for the numerical approximation of (1). In fact, the numerical solution of this kind of problems usually involves the resolution of large linear and nonlinear systems. Since these systems are computationally expensive to solve, the multigrid algorithms provide an efficient way to handle the large systems generated when discretizing the problem. Furthermore, it is natural to look for an algorithm which, in the context of the multigrid approximation, focus on the direct optimization of the energy functional.

The multigrid optimization method (MG/OPT) corresponds to a nonlinear programming adaptation of the *full approximation storage* (FAS) scheme. This approach is proposed, for instance in [8–10], as an effective tool for large scale optimization problems. This algorithm works with different discretization levels of the optimization problem and takes advantage of the coarse problems to generate search directions for the finer problems. Similar approaches have been used for problems involving quasilinear operators, such as the p-Laplace operator (see [11] and the references therein), but to the best of our knowledge, there are no contributions proposing a MG/OPT algorithm for variational inequalities of the second kind involving this kind of operators.

In this paper, we propose and analyze an MG/OPT algorithm to compute the finite element solution of a Huber regularized version of (1). Considering the structure of the optimization problem, specifically the low regularity of the functional, we use a class of preconditioned descent algorithms proposed in [1] as underlying optimization methods or smoothers. Further, the low regularity of the functional prevents us from doing a classical analysis of convergence. Therefore, we perform the convergence analysis of the MG/OPT algorithm by using a mean value theorem for *Bouligand* differentiable functions, which is also applicable for semismooth functions. Finally, we present a comprehensive numerical experimentation focused on the numerical simulation of viscoplastic materials. Specifically, we focus on the flow of these materials through the cross-section of a pipe.

Let us mention that, although the method developed in this article is concerned with variational inequalities of the second kind involving the nonsmooth term $\int_\Omega |\nabla u|\, dx$, the results can be extended to other variational inequalities of the second kind.

The paper is organized as follows: In Section 2, we present several results on generalized differentiability, which will be used to analyze the convergence of the multigrid method. Since the problem is nonsmooth, in Section 3 we propose a local regularization for the objective functional. Further, we present the finite element discretization of the problem. The MG/OPT method is presented in Section 4, whereas the convergence of the algorithm is discussed in Section 5. In Section 6, a brief explanation of the underlying optimization and line search algorithms is presented. In Section 7, we analyze the behavior of the proposed methodology when applied to the numerical simulation of viscoplastic flow. We perform several experiments in order to show the main features of the algorithm. Finally, in Section 8, we outline conclusions on this work and discuss future contributions.

2. Preliminaries on generalized differentiability

This section is devoted to the discussion of several concepts on generalized differentiability. We introduce the *Bouligand* and the slant derivatives of a nonsmooth function, and we discuss the relationship between these two concepts. Further, we present a mean value theorem for *Bouligand* differentiable functions which also holds for semismooth functions.

Definition 2.1. Let X and Y be two normed spaces, D be a nonempty open set in X and $J : D \subset X \to Y$ be a given mapping. For $x \in D$ and $h \in X$, if the limit

$$J'(x)(h) := \lim_{t \to 0^+} \frac{J(x+th) - J(x)}{t}$$

exists, the function is said to be directionally differentiable. Further, $J'(x)(h)$ is the directional derivative of J at x in the direction h.