Computers and Mathematics with Applications 75 (2018) 1181–1190

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On evolutionary Volterra equations with state-dependent delay

Bruno de Andrade *, Giovana Siracusa

Departamento de Matemática, Universidade Federal de Sergipe, São Cristóvão-SE, Brazil

ARTICLE INFO

Article history:
Received 10 April 2017
Received in revised form 4 October 2017
Accepted 30 October 2017
Available online 21 November 2017

Keywords:
Integro-differential equations
State-dependent delay
Viscoelasticity
Fractional differential equations

ABSTRACT

In this work we study topological properties of the solution set of abstract Volterra equations with state-dependent delay, particularly, we ensure that such a set is a nonempty, compact and connected set. As application we consider our abstract results in the framework of integro-differential equations coming from viscoelasticity theory.

1. Introduction

In this work we study some topological properties of the solution set for a class of integro-differential equations with state-dependent delay

$$\begin{cases} u'(t) = \displaystyle\int_0^t a(t-s)Au(s)ds + f\left(t, u_{\rho(t,u_t)}\right), & t \in [0,b], \\ u(0) = \varphi \in \mathfrak{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a closed linear operator defined on a Banach space X, the kernel $a \in L^1_{loc}((0,\infty))$ and the history $u_t : (-\infty, 0] \to X$, given by

$$u_t(\theta) = u(t+\theta),$$

belongs to some abstract phase space $\mathfrak{B}$ described axiomatically. Furthermore, $f : [0,b] \times \mathfrak{B} \to X$ and $\rho : [0,b] \times \mathfrak{B} \to (-\infty, b]$ are given functions. From the mathematical point of view, we are motivated by elegance and simplicity that evolutionary integro-differential equations of the type (1.1) provides to problems in mathematical physics.

As typical application of (1.1) we consider the problem

$$\begin{cases} u_t(t,x) = \displaystyle\int_0^t da(s)u_{xx}(t-s,x) + h\big(t,x,u(t-\sigma(\|u(t,x_0)\|),x)\big), \ t \geq 0, \ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t > 0, \\ u(t,x) = \varphi(t,x), \quad t \leq 0, \quad x \in [0,\pi], \end{cases}$$

* Corresponding author.
E-mail addresses: bruno00luis@gmail.com (B. de Andrade), gisiracusa@gmail.com (G. Siracusa).

https://doi.org/10.1016/j.camwa.2017.10.039
0898-1221/

where $x_0 \in (0, \pi)$ is fixed, $a : [0, \infty) \to (0, \infty)$ is a function of bounded variation on each compact interval $J = [0, T]$, $T > 0$, with $a(0) = 0$, and

$$\sigma : [0, \infty) \to [0, \infty)$$

is a continuous function. This type of equations has been the subject of many research papers in the last years since it has applications in such different fields as the theory of viscoelastic materials, thermodynamics, electrodynamics and population biology, cf. [1–8] and references therein.

In the literature problem (1.1) has been studied by several authors. For example, Agarwal et al. [9] ensure sufficient conditions to existence of mild solutions to the particular case of (1.1) where

$$a(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}, \; t > 0,$$

$\alpha \in (1, 2)$ and Γ means the Gamma function. This kind of kernel is called of *fractional time order* and has great importance, for example, in the theory of heat conduction in materials with memory (see [4,10]). The same problem was also studied by Benchohra et al. [11]. However, they assumed that A is the generator of a certain solution.

The study of the topological structure of solution set of differential equations dates back to the beginning of the 20's when H. Kneser (see [12]) proved that the Peano existence theorem could be reformulated to ensure that the solution set of a ODE is, beyond nonempty, a compact and connected set. This property is known in the literature as the Kneser's property. Almost 20 years later, N. Aronszajn (see [13]) improved the Kneser theorem showing that the set of all solutions of a ODE is an R_δ-set, that is, an intersection of a decreasing sequence of compact absolute retracts sets. Evidently the Aronszajn theorem had a large impact on qualitative theory of differential equations and due to this the study of topological structure of the solution set of differential equations has drawn attention of researchers in the last years, see for instance [2,14–18] and references therein. On the other hand, it is well known that the characterization of the fixed points set of some operators is a very useful result in the study of differential equations. In this sense, a powerful information on fixed points set was given by F. Browder and C. Gupta. This result gave rise to so called Browder–Gupta method. In this work a slight generalization of the above mentioned result, which can be found at [17] (see also Theorem 2.1), will be very useful for our purpose.

We remark that research on differential equations with state-dependent delay has received great attention in the last years. We just mention a few of these works. Cuevas et al. [19] study existence of mild solutions to nonautonomous impulsive neutral differential equations with state-dependent delay using the Leray–Schauder alternative theorem and Krasnoselskii's fixed point theorem. In [20], Fengde et al. consider the existence of periodic solutions in a population model which incorporates food limitations, periodic environment and regulation delays. The mathematical techniques used by them are based on the continuation theorem from Gaines and Mawhin's coincidence degree theory. In the works [21,22], the authors consider existence of periodic solutions to state-dependent delay equations and Hartung [23] deals with linearized stability in periodic functional differential equations with state-dependent delays. Stimulated by these works in this paper we will investigate further the corresponding problem in the situation of integro-differential equations.

The organization of the paper is as follows. In Section 2 we provide the necessary definitions and preliminary results. Particularly, we review some of the standard properties of abstract phase spaces, solution operators and some concepts and tools of Algebraic Topology. The main result of this paper is contained in Section 3. Basically it says that, under suitable conditions, the set $\mathcal{S}$ formed by the mild solutions of the problem (1.1) is a R_δ-set. Particularly, it is a nonempty compact and connected space. Furthermore, it is acyclic with respect to the Čech homology functor which means that from the point of view of Algebraic Topology, it is equivalent to a point, in the sense that it has the same homology groups as one point set.

2. Preliminaries and basic results

Let X be a Banach space. In this work, we denote by $\mathcal{B}(X)$ the space of bounded linear operators from X into X endowed with the norm of operators, and for a linear operator $A : D(A) \subset X \to X$ the notation $\rho(A)$ stands for the resolvent set of A. In the remainder of this section we provide the definitions and preliminaries results to be used in this work.

We start with the axiomatic definition of the phase space $\mathfrak{B}$ using ideas and notations developed in [24]. More precisely $\mathfrak{B}$ will denote the vector space of functions defined from $(-\infty, 0]$ into X endowed with a seminorm denoted $\|\cdot\|_{\mathfrak{B}}$ and such that the following conditions hold:

(**A**) If $w : (-\infty, \nu + b) \to X$, $b > 0$, $\nu \in \mathbb{R}$, is continuous on $[\nu, \nu + b)$ and $w_\nu \in \mathfrak{B}$, then for every $t \in [\nu, \nu + b)$ the following conditions hold:

(i) w_t belongs to $\mathfrak{B}$.
(ii) $\|w(t)\| \le H\|w_t\|_{\mathfrak{B}}$.
(iii) $\|w_t\|_{\mathfrak{B}} \le K(t-\nu)\sup\{\|w(s)\| : \nu \le s \le t\} + M(t-\nu)\|w_\nu\|_{\mathfrak{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and H, K, M are independent of $w(\cdot)$.

(**A1**) For the function $w(\cdot)$ in (**A**), the function $t \to w_t$ is continuous from $[\nu, \nu + b)$ into $\mathfrak{B}$.

(**B**) The space $\mathfrak{B}$ is complete.