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Lie group method for solving viscous barotropic vorticity equation in ocean climate models

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ABSTRACT

In studying the problem of the nonlinear viscous barotropic non-divergent vorticity equation on f- and β- planes, the method of Lie group has been applied. The method reduces the number of independent variables by one, and consequently, for the case of three independent variables we applied the method successively twice and the nonlinear partial differential equation reduces to ordinary differential equation. Investigation of exact solutions of the viscous barotropic non-divergent vorticity equation on f- and β- planes, via the application of Lie group, provides large classes of new exact solutions which include both Rossby and Rossby–Haurwitz waves as special cases. Also, The Lie symmetries of the viscous barotropic non-divergent vorticity equation with two parameters F and β, are determined. The possible reductions of the viscous barotropic vorticity equation with two parameters F and β have been investigated by means of one- dimensional Lie subalgebras.

1. Introduction

It is very interesting to obtain the exact solutions of nonlinear partial differential equation (PDE). The exact solutions may well describe various phenomena in our life. Also, these exact solutions when they exist can help to understand the dynamical processes that are modeled by the corresponding nonlinear evolution equations (NLEEs). Ocean is a major component of the earth climate system, since ocean has large heat storage. Roughly, three meters of sea water has about the same heat capacity as the whole atmospheric column above it, ocean heat storage modulates diurnal and seasonal cycles and climate variations. Waves in fluids can be simulated with mathematical models based on the equations for conservation of mass, momentum and energy.

These equations are the starting point for modeling the propagation of Rossby waves and gravity waves in the atmosphere and ocean. The derivation of the basic equations can be found in any fluid dynamics text such as Batchelor (1967), Pond and Pickard (1983), Pedlosky (1992), Kundu and Cohen (2004), Marshall and plumb (2008), Vallis (2011) and Talley, Pickard, Emery and Swift (2011) [1–8]. In geophysical fluid dynamics, the effects of the earth's rotation and the density-stratification of the basic flow are taken into consideration. There are various types of waves in geophysical flows in the atmosphere and ocean [9,10]. These include internal gravity waves which result from the effects of gravity and the buoyancy of the fluid flow, and Rossby waves which result from the effects of the earth's rotation. The driving mechanism of Rossby waves is the interaction of the flow with meridional variations of the Coriolis parameter f. Considering $f = 2\Omega \sin\varphi$ [3–5], which, due to spherical shape of the Earth, depends on latitude φ. We can expand f as a Taylor series about the reference latitude φ_0, and approximate it by the first two terms. Hence, $f = f_0 + \beta y$ where $f_0 = 2\Omega \sin\varphi_0$, $\beta = (2\Omega/r) \cos\varphi_0$, r is the earth's radius, and Ω the earth's angular velocity. Typical mid-latitude values are $f_0 = 8 \times 10^{-5}\ \mathrm{s}^{-1}$ and $\beta = 1 \times 10^{-11}\ \mathrm{m}^{-1}\ \mathrm{s}^{-1}$. From a

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mathematical perspective, Rossby waves can be studied in 2 D using Cartesian coordinates in a plane tangent to the surface of the earth, where x for the West–East (zonal) coordinate and y for the South–North (meridional) coordinate, the domain in this case should be assumed to be infinite allowing the waves to propagate indefinitely southwards. If this approximation of f is applied in a Cartesian framework, this framework is called the f-plane if the β term is neglected, and the β- plane if the beta term is retained.

1.1. Barotropic ocean climate model

The viscous barotropic non-divergent vorticity equation model is very important in the field of meteorological science and applied mathematics. Many scientists pay attention to the research of numerical methods of this equation. Most of them mainly apply the finite difference methods [11–13]. The analytical solutions of barotropic vorticity equation early obtained by Ekman (1905), Rossby (1937, 1938), Sverdrup (1947), Stommel (1948), Munk (1950), Charney (1955) and Morgan (1956) [14–21], who among them identified the main dynamical features of the large-scale ocean circulation, previous models details have either considered linear frictional model, e.g., Munk, or nonlinear inviscid models, e.g., Charney and Morgan. In (1963) Moore constructed a model of the general circulation in an ocean basin which includes viscous effects and inertial effects through an Oseen linearization [22].

The nonlinear viscous barotropic non-divergent vorticity equation in stream function form with β-plane approximation reads [3,8]

$$\nabla^2\psi_t - \psi_y\nabla^2\psi_x + \psi_x\nabla^2\psi_y + \beta\,\psi_x - \nu\,\nabla^4\psi = 0, \tag{1.1}$$

where ψ is the steam function, the velocity field $\{u\,,\ v\}$ is linked to stream functions by $u = -\psi_y$ and $v = \psi_x$, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, ν is the kinematic viscosity, $\beta = \frac{df}{dy}$ is the meridional change of the Coriolis parameter and $\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

Eq. (1.1) describes the fluid motion on a β-plane with viscosity effect in terms of the stream function ψ and the term $\xi = \nabla^2\psi$ which is known as the relative vorticity, measures the amount of rotation that the fluid undergoes in the $x - y$ plane and is given by

$$\xi = \vec{e}_z \cdot \nabla \times \vec{u} = \nabla^2\psi, \tag{1.2}$$

where $\vec{e}_z$ is the unit vector in the z direction.

By substituting (1.2) in (1.1), we get

$$\xi_t + \psi_x\,\xi_y - \psi_y\,\xi_x + \beta\,\psi_x - \upsilon\,\nabla^2\xi = 0. \tag{1.3}$$

We notice that value of the Coriolis frequency f_0 has dropped entirely out of the equation; this means that any of the results discussed in this paper are independent of f_0 and hence independent of the Rossby number. Eq. (1.3) can be written as

$$\frac{D\xi}{Dt} = -\beta\,v + \upsilon\,\nabla^2\xi. \tag{1.4}$$

If there is no rotation, i.e. $\beta = 0$, (1.4) resembles the energy conservation equation. In the context of rotating flows is called ξ the potential vorticity or the relative vorticity. The total vorticity $\eta = \xi + f_0 + \beta\,y$, is called the absolute vorticity. Eq. (1.4) represents the conservation of potential vorticity.

The generalization of (1.3), it has been successfully used both for theoretical consideration and practical numerical weather predication since it is capable of describing some prominent features of mid-latitude weather phenomena such as the well known Rossby waves and blocking regimes. In non-dimensional form it reads [3]

$$\xi_t - F\,\psi_t + \psi_x\,\xi_y - \psi_y\,\xi_x + \beta\,\psi_x - \upsilon\,\nabla^2\xi = 0, \tag{1.5}$$

where F represents the ratio of characteristics length scale to the Rossby radius of deformation. The Lie symmetry of (1.5) arises since different values of F and β leads to different Lie symmetry properties of (1.5), there are only three combination of the values of two parameter, given by $F = 0,\ \beta = 0$; $F = 0,\ \beta \neq 0$ and $F \neq 0$. The first combination leads to the vorticity form of Euler's equations, which has been discussed in [23]. The second combination leads to the barotropic vorticity equation, which has been completely discussed here. For $F \neq 0$ we can set $\beta = 0$, then F can be scaled to ± 1. For this consideration, we recomputed the symmetries of (1.5) for the case $F \neq 0$ and β arbitrary. Hence, we investigate interesting group invariant reductions for (1.5) based on the obtained symmetries.

The boundary condition plays an important role in conception of geophysical fluid dynamics. With 2D ocean models, since the water coming in contact with a solid boundary, therefore it subjects to a no-slip condition, in practice the no slip condition very difficult to resolve in ocean models, so many different approximation applied to boundary condition such that free slip ($\xi = 0$) on the boundary where is possible and hyper slip ($\partial_n\xi = 0$) which has boundary layers that are similar in nature to free slip boundary layers.

Munk [18] introduced the solution for (1.3) by considering $\beta\,v \approx \nu\,\xi_{xx}$, Stommel [17] introduced the solution for (1.3) by considering $\beta\,\nu \approx -r\,\xi$. Finally Charney [19] introduced the solution for (1.3) by considering $\psi_{xx} + \beta\,y \approx -\frac{\beta}{U_0}\psi$.

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