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Radial Basis Function generated Finite Differences for option pricing problems

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ABSTRACT

In this paper we present a numerical method to price options based on Radial Basis Function generated Finite Differences (RBF-FD) in space and the Backward Differentiation Formula of order 2 (BDF-2) in time. We use Gaussian RBFs that depend on a shape parameter ε. The choice of this parameter is crucial for the performance of the method. We chose ε as const $\cdot\, h^{-1}$ and we derive suitable values of the constant for different stencil sizes in 1D and 2D. This constant is independent of the problem parameters such as the volatilities of the underlying assets and the interest rate in the market. In the literature on option pricing with RBF-FD, a constant value of the shape parameter is used. We show that this always leads to ill-conditioning for decreasing h, whereas our proposed method avoids such ill-conditioning. We present numerical results for problems in 1D, 2D, and 3D demonstrating the useful features of our method such as discretization sparsity, flexibility in node placement, and easy dimensional extendability, which provide high computational efficiency and accuracy.

1. Introduction

Calibration and pricing of financial instruments is something that is going on daily in the financial industry. In most cases there are no analytical solutions available and therefore numerical methods have to be used for this purpose. Hence, accurate and efficient methods for this kind of problems are of utmost importance. In this paper our focus is on developing methods for pricing of multi-asset options, i.e. when an option is issued on several underlying assets. This leads to a high-dimensional problem which is numerically very challenging to solve.

We start by considering the Black–Scholes–Merton model with a risk free asset B and a risky asset S that follow the dynamics

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW(t), \end{aligned} \tag{1}$$

where t is time, r is the interest rate, μ is the drift and σ is the volatility of S, and W is the Wiener process. A European type option issued on S, maturing at time T, with a payoff function $g(S(T))$ can be priced from

$$u(S(t), t) = e^{-r(T-t)}\mathbb{E}^{Q}\left[g(S(T))\right], \tag{2}$$

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where $\mathbb{E}^Q[\cdot]$ denotes the expected value under the risk-neutral measure Q. In [1] and [2] it was independently shown that the price of an option can also be obtained by solving

$$\frac{\partial u}{\partial t}+rs\frac{\partial u}{\partial s}+\frac{1}{2}s^2\sigma^2\frac{\partial^2 u}{\partial s^2}-ru=0, \qquad u(s,T)=g(s). \tag{3}$$

Eq. (3) is a parabolic partial differential equation (PDE), which can be solved backward in time using e.g. Finite Differences (FD) in space [3–6].

Next, we turn to multi-asset options that depend on D underlying assets $S_d(t)$, $d=1,\dots,D$. The multi-dimensional analogue to (1) is

$$\begin{aligned} dB(t) &= rB(t)dt,\\ dS_1(t) &= \mu_1 S_1(t)dt+\sigma_1 S_1(t)dW_1(t),\\ dS_2(t) &= \mu_2 S_2(t)dt+\sigma_2 S_2(t)dW_2(t),\\ &\vdots\\ dS_D(t) &= \mu_D S_D(t)dt+\sigma_D S_D(t)dW_D(t), \end{aligned} \tag{4}$$

where the Wiener processes are correlated such that $dW_i(t)dW_j(t)=\rho_{i,j}dt$. In this high-dimensional setting, an option issued on the assets (4) with payoff function $g(S_1(T),\dots,S_D(T))$ can be priced from

$$u(S_1(t),\dots,S_D(t),t)=e^{-r(T-t)}\mathbb{E}_t^Q[g(S_1(T),\dots,S_D(T))].$$

The corresponding high-dimensional Black–Scholes–Merton equation reads

$$\frac{\partial u}{\partial t}+r\sum_i^D s_i\frac{\partial u}{\partial s_i}+\frac{1}{2}\sum_{i,j}^D\rho_{i,j}\sigma_i\sigma_j s_i s_j\frac{\partial^2 u}{\partial s_i\partial s_j}-ru\equiv\frac{\partial u}{\partial t}+\mathcal{L}u=0, \qquad u(s_1,s_2,\dots,s_D,T)=g(s_1,s_2,\dots,s_D). \tag{5}$$

Since FD methods are generally discretized on tensor product grids of 1D Cartesian grids, the number of degrees of freedom grows exponentially in the number of dimensions D—the so-called *curse of dimensionality*. Traditionally, Monte-Carlo methods have been the only way to price options in dimensions larger than approximately 5.

Global radial basis functions (RBFs) approximation methods are mesh-free, meaning that they are flexible with respect to the geometry of the computational domain. Hence, we are no longer restricted to Cartesian grids, and we can more freely place nodes where they are needed for accuracy reasons. Moreover, the methods are not more complicated for high-dimensional problems than in lower dimensions, since the only geometrical property that is used is the pairwise distance between points. Finally, for smooth functions, approximations with smooth RBFs can give spectral convergence. When RBFs are used, the space is discretized using N nodes $\mathbf{s}^{(i)}$ and the solution is approximated by

$$u(\mathbf{s},t)\approx\sum_{i=1}^N\lambda_i(t)\phi(\|\mathbf{s}-\mathbf{s}^{(i)}\|),\ k=1,2,\dots,N,$$

where $\phi(r)$ is a radial basis function. Possible radial basis functions are eg. Gaussian ($e^{-(\varepsilon r)^2}$), inverse quadratic ($1/(1+(\varepsilon r)^2)$), multiquadric ($\sqrt{1+(\varepsilon r)^2}$) and inverse multiquadric ($1/\sqrt{1+(\varepsilon r)^2}$), where ε is a shape parameter that determines the width of the basis function, see e.g. [7].

To sparsify the linear systems of equations, various techniques to localize the RBFs are possible such as RBF Partition of Unity Methods (RBF-PUM) [8–11], where the computational domain is partitioned into subdomains. In this paper we consider an even more localized strategy, RBF generated Finite Difference (RBF-FD) [12–18]. In RBF-FD methods, the finite difference weights in the computational stencils are computed from RBFs rather than from the monomials $\{1,x,x^2,\dots\}$ which are used in standard finite differences.

We would like to have a solution method for (5) that yields the same type of sparsity structure as FD but is as easy to employ in higher dimensions as the RBF method. By deriving finite difference approximations based on radial basis function approximations we expect to achieve these properties. RBF-FD for option pricing has previously been used in e.g. [19–23] for various types of options. In all these references, 1D and in some cases 2D problems are considered with uniform node layouts and (very) small stencils, which severely restricts the potential usefulness of the method. Moreover, no analysis or suggestions on the choice of the shape parameter ε is provided which is important for the method to be used in practice.

In this paper we propose an RBF-FD method for option pricing that can be used with non-uniform node layouts on suitably shaped domains and in any number of dimensions. We provide numerical examples in up to three dimensions. Also, we analyze our method with respect to accuracy and choice of shape parameter ε. From this analysis and numerical experiments, we propose a way of choosing this important parameter.

The article is organized as follows. In Section 2 we describe the spatial and temporal discretization as well as the treatment of the open boundary problem for the American options. The spatial error is analyzed for a 1D problem in Section 3 and numerical experiments are presented in Section 4. Finally in Section 5 we summarize and draw some conclusions.