ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Conservation laws of $(3+\alpha)$-dimensional time-fractional diffusion equation

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ARTICLE INFO

Article history:
Received 21 February 2017
Received in revised form 27 September 2017
Accepted 7 October 2017
Available online xxxx

Keywords:
Fractional symmetry
Fractional conservation laws
Fractional diffusion equation
Laplace operator
Noether's operator

ABSTRACT

The concept of Lie–Backlund symmetry plays a fundamental role in applied mathematics. It is clear that in order to find conservation laws for a given partial differential equations (PDEs) or fractional differential equations (FDEs) by using Lagrangian function, firstly, we need to obtain the symmetries of the considered equation.

Fractional derivation is an efficient tool for interpretation of mathematical methods. Many applications of fractional calculus can be found in various fields of sciences as physics (classic, quantum mechanics and thermodynamics), biology, economics, engineering and etc. So in this paper, we present some effective application of fractional derivatives such as fractional symmetries and fractional conservation laws by fractional calculations. In the sequel, we obtain our results in order to find conservation laws of the time-fractional equation in some special cases.

1. Introduction

As we know conservation laws is one of the substantial concepts in physics and mathematics. In this study, conservation laws, for FDEs [1–9], are derived by symmetry group technique.

About three centuries ago for finding conservation laws of the considered equation, for the first time, derivatives of non-integer order ($\alpha=\frac{1}{2}$) was introduced by Leibniz. Recently this methodology is developed for PDEs for finding exact solutions, symmetries, conservation laws and etc.

The concept of conservation laws is given in [10–14]. Furthermore, it has been shown that how we can make conservation laws by using Lie–Backlund symmetry generators. In a number of recent papers, conservation laws by using fractional symmetry for FDEs is developed [15,16]. In this method, the existence of Lagrangian is not necessary. In fact by formal Lagrangian, we yield the components of conservation laws [13].

The classical Lie point symmetry has been investigated for the diffusion equation of the $(3+\alpha)$ dimensional of order $\alpha\in(0,2)$

$$D_t^{\alpha}u=(f(u)u_x)_x+(g(u)u_y)_y+(h(u)u_z)_z, \tag{1}$$

in the sense of Riemann–Liouville derivative. Furthermore, we describe our methodology for calculating of conservation laws for FDEs. In the sequel, as an example and application of the proposed model, we present the calculation of the conservation laws. In the process of computing the symmetry and conservation laws, we use fractional calculating regularly [17–19].

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https://doi.org/10.1016/j.camwa.2017.10.001

The present paper is organized as follows. In section two, we illustrate some fractional results for the considered FDE (1). Third chapter is devoted to find symmetry operators of Eq. (1) in general and special forms. In the fourth chapter, we give a comprehensive analysis of the equation including application of symmetry analysis and conservation laws.

2. Fractional calculus on the Eq. (1)

Left sided and right sided time-fractional integral of order $n-\alpha$ are defined as follows.

$$({}_0I_t^{n-\alpha})(t,x,y,z)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u(\mu,x,y,z)}{(t-\mu)^{\alpha+1-n}}d\mu \tag{2}$$

$$({}_tI_T^{n-\alpha})(t,x,y,z)=\frac{1}{\Gamma(n-\alpha)}\int_t^T\frac{u(\mu,x,y,z)}{(\mu-t)^{\alpha+1-n}}d\mu. \tag{3}$$

Left and right derivatives of Riemann–Liouville and Caputo time-fractional derivatives is described by:

$${}_0D_t^{\alpha}u=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t\frac{u(\mu,x,y,z)}{(\mu-t)^{\alpha+1-n}}d\mu, \tag{4}$$

$${}_tD_T^{\alpha}u=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_t^T\frac{u(\mu,x,y,z)}{(\mu-t)^{\alpha+1-n}}d\mu, \tag{5}$$

$$({}^CD_t^{\alpha})(t,x,y,z)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{D^nu(\mu,x,y,z)}{(\mu-t)^{\alpha+1-n}}d\mu, \tag{6}$$

$${}_t^CD_T^{\alpha}u=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^T\frac{D_{\mu}^n(\mu,x,y,z)}{(\mu-t)^{\alpha+1-n}}d\mu. \tag{7}$$

So, we have

$${}_0D_t^{\alpha}u=D_t^n({}_0I_t^{n-\alpha}u),\qquad {}_0^CD_t^{\alpha}u={}_0I_t^{n-\alpha}(D_t^nu).$$

In the sequel, four time-fractional generalizations of Eq. (1) are considered.

First let us that in the resumption, the classical diffusion equation can be written as $u_t=C[U]$, where

$$C[U]=(f(u)u_x)_x+(g(u)u_y)_y+(h(u)u_z)_z,$$

four time-fractional generalized to Eq. (1) are

$$u_t=I_t^{\alpha}C[U]=I_t^{\alpha}\left[(f(u)u_x)_x+(g(u)u_y)_y+(h(u)u_z)_z\right], \tag{8}$$

$$u_t=D_t^{1-\alpha}C[U]=D_t^{1-\alpha}\left[(f(u)u_x)_x+(g(u)u_y)_y+(h(u)u_z)_z\right], \tag{9}$$

$$u_t=C[I_t^{\alpha}U]=f_uI_t^{\alpha}u_x^2+fI_t^{\alpha}u_{xx}+g_uI_t^{\alpha}u_y^2+gI_t^{\alpha}u_{yy}+h_uI_t^{\alpha}u_z^2+hI_t^{\alpha}u_{zz}, \tag{10}$$

$$\begin{aligned}u_t=C[D_t^{1-\alpha}U]&=f_uD_t^{1-\alpha}u_x^2+fD_t^{1-\alpha}u_{xx}+g_uD_t^{1-\alpha}u_y^2+gD_t^{1-\alpha}u_{yy}\\&\quad+h_uD_t^{1-\alpha}u_z^2+hD_t^{1-\alpha}u_{zz}.\end{aligned} \tag{11}$$

where I_t^{α} is the left-sided time-fractional integral of order α and $D_t^{1-\alpha}$ is the left-sided time-fractional derivative of order $1-\alpha$ of the Riemann–Liouville type.

Note that since $D_t^{\alpha}I_t^{\alpha}u=u$, Eq. (8) , can be rewritten as

$$D_t^{\alpha+1}u=D_t^{\alpha}u_t=(f(u)u_x)_x+(g(u)u_y)_y+(h(u)u_z)_z. \tag{12}$$

Substituting $D_t^{1-\alpha}=I_t^{\alpha-1}$ and $I_t^{\alpha-1}u_t={}^CD_t^{\alpha}u$ into Eq. (9) yields

$${}^CD_t^{\alpha}u_t=(f(u)u_x)_x+(g(u)u_y)_y+(h(u)u_z)_z. \tag{13}$$

Taking $v=I_t^{\alpha}u$, $\left(I_t^{\alpha}u_{xx}=v_{xx}, I_t^{\alpha}u_x^2=v_x^2, u_t=D_t^{1+\alpha}v\right)$ from Eq. (10) we have

$$D_t^{1+\alpha}v_t=(f(v)v_x)_x+(g(v)v_y)_y+(h(v)v_z)_z,$$