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Existence of ground states for a Kirchhoff type problem without 4-superlinear condition☆

Ke Wu

Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650500, PR China

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ABSTRACT

We consider the existence of ground state solutions for the Kirchhoff type problem

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx)\triangle u+V(x)u=|u|^{p-2}u, \quad x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\end{cases}$$

where $a, b > 0$, $N = 1, 2, 3$ and $2 < p < 2^*$. Here we are interested in the case that $2 < p \le 4$ since the existence of ground state for $4 < p \le 2^*$ is easily obtained by a standard variational argument. Our method is based on a Pohožaev type identity.

1. Introduction and main result

Consider the existence of ground state solutions for Kirchhoff type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx)\triangle u+V(x)u=|u|^{p-2}u,\ x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\end{cases} \tag{1.1}$$

where $a, b > 0$, $N = 1, 2, 3$ and $2 < p < 2^*$. This is a nonlocal problem as the appearance of the term $(b\int_{\mathbb{R}^N}|\nabla u|^2dx)\triangle u$, which implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties which make the study of nonlocal problems like (1.1) particularly interesting.

Such problems like (1.1) arise in some biological systems and various branches of mathematical physics. Indeed, it is related to the stationary analogue of the equation

$$\rho\frac{\partial^2u}{\partial t^2}-(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx)\frac{\partial^2u}{\partial x^2}=0 \tag{1.2}$$

proposed by Kirchhoff in [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, where $\frac{P_0}{h}$ denotes the initial tension and $\frac{E}{2L}$ is related to the intrinsic properties of the string, such as Young's modulus. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. We also point out that such problems may describe a process of some biological systems dependent on the average of itself, such as the density of population (see e.g. [2–4]).

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E-mail address: wuke2002@126.com.

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In addition, problem (1.1) includes the well-known Nonlinear Schrödinger Equation(NSE)

$$-\triangle u+V(x)u=|u|^{p-2}u,\ \ x\in\mathbb{R}^N$$

corresponding to the special case $a=1$ and $b=0$. The NSE arises in various fields physics, such as nonlinear optics, quantum mechanics and the classical approximation in statistical mechanics. In particular, it can be used to describe the evolution of a free non-relativistic quantum particle in quantum mechanics. Besides, in nonlinear optics, one can reduce the Maxwell's equations to the NSE (see [5]). For more detailed information dealing with applications of problem (1.1) and the NSE, we refer readers to [6–12] and the references therein.

From the viewpoint of mathematics, Kirchhoff type problems began to attract much attention after J.L. Lions [13] proposed an abstract framework to them. In particular, the following elliptic problem

$$\begin{cases}-(a+b\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2dx)\triangle u+V(x)u=f(x,u),\ \ x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\end{cases}\tag{1.3}$$

has been studied by several authors in recent years, for example, see [14–23].

A typical method dealing with (1.3) is the application of the mountain pass theorem that allows to prove the existence and multiplicity results under the 4-superlinear condition:

(F) $f(x,u)u\geq 4F(x,u)$ for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$ and $\lim_{u\to\infty}\frac{F(x,u)}{u^4}=+\infty$ uniformly in $x\in\mathbb{R}^N$, where $F(x,u)=\int_0^u f(x,t)dt$.

In fact, if the assumption (F) is satisfied, then the (PS) sequences corresponding to (1.3) are bounded and their weak limits are always nontrivial solutions under some further conditions on the nonlinear term $f(x,u)$ and the potential $V(x)$ (see e.g. [18]). However, this method does not work for both 4-linear and 4-sublinear problems. A classical example of this kind of problems may be Eq. (1.1) with $2<p\leq 4$. Without the aid of condition (F), it seems that one cannot obtain the boundedness of a (PS) sequence, let alone its convergence. We also remark that, different from the case that 4-superlinear problems, there are very few results on the existence of solutions for (1.3) without the condition (F). Recently, Li et al. [14] concluded the existence of ground state solutions for (1.1) with $N=3$, $3<p<6$, $\nabla V(x)\cdot x\in L^\infty(\mathbb{R}^3)\bigcup L^{\frac{3}{2}}(\mathbb{R}^3)$, $V(x)-\nabla V(x)\cdot x\geq 0$ for a.e. $x\in\mathbb{R}^3$ and some further assumptions on $V(x)$.

In this paper, our goal is trying to find the ground state solutions for (1.1) with $2<p\leq 4$. By introducing some new techniques, we prove the compactness of a special (PS) sequence (see Section 3). In addition, we do not need the condition $V(x)-\nabla V(x)\cdot x\geq 0$ for a.e. $x\in\mathbb{R}^3$ which seems essential in [14]. More precisely, we obtain the existence of ground state solutions for (1.1) under the following assumptions on the potential $V(x)$.

(V_1) $V\in C^1(\mathbb{R}^N)$ satisfies $0<V_0\leq V(x)\leq\lim_{|x|\to\infty}V(x)=V_\infty<+\infty$;

(V_2) $\nabla V(x)\cdot x\in L^\infty(\mathbb{R}^N)$ and the function $t\mapsto t^{3-p}\nabla V(tx)\cdot x$ is nonincreasing on $(0,\infty)$ for any $x\in\mathbb{R}^N$.

Our main result is the following theorem.

Theorem 1.1. *Let $p_N=2$ if $N=1,2$ and $p_N=3$ if $N=3$. Assume that $p_N<p\leq 4$ and the conditions (V_1), (V_2) hold. Then (1.1) has a ground state solution (a solution whose energy is minimal among the set of nontrivial solutions to (1.1)).*

We give some comments on the conditions (V_1) and (V_2). It is known that, under the hypothesis (V_1), $V(x)$ is called a well potential which is quite usual for the existence problem. Although a positive lower bound of the potential is assumed in (V_1), our main result presented here seems still true if replace (V_1) with (V_2) used in [24]. The condition $\nabla V(x)\cdot x\in L^\infty(\mathbb{R}^N)$ may not be very restrictive in the sense that (V_1) implies that $\lim_{|x|\to\infty}|\nabla V(x)|=0$. The monotonicity hypothesis on V in (V_2) is technical. However, it seems essential for our arguments, see Lemma 2.2. Clearly, a trivial example satisfying (V_1) and (V_2) is the constant functions. We also remark that the following assumption (V_3) has been used in [24].

(V_3) $V\in C^1(\mathbb{R}^3,\mathbb{R})$ and there exists a positive constant $A<a$ such that

$$|\nabla V(x)\cdot x|\leq\frac{A}{2|x|^2},\ \ \forall x\in\mathbb{R}^3\setminus\{0\}.$$

Obviously, (V_3) implies that $\nabla V(x)\cdot x\in L^\infty(\mathbb{R}^N)$ has a strict restriction. In this direction, our condition (V_2) seems more relax. However, we are not sure if all functions satisfying (V_3) belong to the class defined in (V_2).

Remark 1.2. Recall that $p>2$. By (V_2), we have $\nabla V(x)\cdot x\geq t^{2-p}\nabla V(tx)\cdot(tx)$ for any $t>1$ and $x\in\mathbb{R}^N$. This implies that

$$\nabla V(x)\cdot x\geq\lim_{t\to+\infty}t^{2-p}\nabla V(tx)\cdot(tx)=0\tag{1.4}$$

for any $x\in\mathbb{R}^N$. Moreover, let $f(s)=\nabla V(tx)\cdot(tx)s^{p-2}-(p-2)V(stx)$ for $s\in[0,1]$. By (V_2), we have

$$\begin{aligned}f'(s)&=(p-2)\nabla V(tx)\cdot(tx)s^{p-3}-(p-2)\nabla V(stx)\cdot(tx)\\&\leq(p-2)\nabla V(stx)\cdot(tx)-(p-2)\nabla V(stx)\cdot(tx)=0.\end{aligned}$$

This implies that $f(s)$ is decreasing. Then $f(1)\leq f(0)$, that is,

$$\nabla V(tx)\cdot(tx)-(p-2)V(tx)\leq-(p-2)V(0)\leq 0.\tag{1.5}$$

Therefore, $t^{2-p}V(tx)$ is nonincreasing on $(0,+\infty)$ with respect to t.

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