Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The reconstruction of a time-dependent source from a surface measurement for full Maxwell's equations by means of the potential field method

T. Kang [a], K. Van Bockstal [b,*], R. Wang [a]

[a] *Department of Applied Mathematics, School of Sciences, Communication University of China, China*

[b] *Department of Mathematical Analysis, Research group of Numerical Analysis and Mathematical Modeling (NaM²), Ghent University, Galglaan 2 - S22, Gent 9000, Belgium*

ARTICLE INFO

Article history:
Received 2 March 2017
Received in revised form 7 July 2017
Accepted 10 October 2017
Available online xxxx

Keywords:
Maxwell's equations
Inverse source problem
$\boldsymbol{A}$-ϕ method
Reconstruction
Time discretization

ABSTRACT

This paper is devoted to the study of an inverse source problem governed by full Maxwell's equations by means of the potential field method (the $\boldsymbol{A}$-ϕ method). The source term is assumed to be separable in time and space, in which the unknown part is solely time-dependent and is recovered from a surface measurement. We prove that the solution to the inverse problem based on the $\boldsymbol{A}$-ϕ formulation is existing and unique. We suggest a constructive scheme for approximating the solution and discuss its convergence. Finally, a few examples are presented to verify the theoretical results.

1. Introduction

In this contribution, an inverse source problem governed by the full Maxwell's equations is studied by means of the potential field method (the $\boldsymbol{A}$-ϕ method). A bounded, simply-connected and convex polyhedron $\Omega \subset \mathbb{R}^3$ with connected boundary $\partial\Omega$ is considered. The outer normal vector associated with the boundary is denoted by $\boldsymbol{n}$. The current density $\boldsymbol{J}$ satisfies Ohm's law such that $\boldsymbol{J} = \sigma\boldsymbol{E} + \boldsymbol{J}_a$, where the vector function $\boldsymbol{J}_a$ describes possible current sources in Ω. Electromagnetic fields are described by the full linear Maxwell's equations, i.e.

$$\begin{cases} \partial_t \boldsymbol{D}(\boldsymbol{x}, t) - (\nabla \times \boldsymbol{H})(\boldsymbol{x}, t) = -\boldsymbol{J}, & (\boldsymbol{x}, t) \in \Omega \times (0, T] \\ \partial_t \boldsymbol{B}(\boldsymbol{x}, t) + (\nabla \times \boldsymbol{E})(\boldsymbol{x}, t) = \boldsymbol{0}, & (\boldsymbol{x}, t) \in \Omega \times (0, T] \end{cases}$$

or in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ as

$$\begin{cases} \varepsilon(\boldsymbol{x})\partial_t \boldsymbol{E}(\boldsymbol{x}, t) - (\nabla \times \boldsymbol{H})(\boldsymbol{x}, t) = -\left(\sigma(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{J}_a(\boldsymbol{x}, t)\right), & (\boldsymbol{x}, t) \in \Omega \times (0, T] \\ \mu(\boldsymbol{x})\partial_t \boldsymbol{H}(\boldsymbol{x}, t) + (\nabla \times \boldsymbol{E})(\boldsymbol{x}, t) = \boldsymbol{0}, & (\boldsymbol{x}, t) \in \Omega \times (0, T] \end{cases} \tag{1.1}$$

where $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are the electric displacement field, electric field, magnetic induction and the magnetic field, σ, ε and μ denote the electric conductivity, the electric permittivity and the magnetic permeability of the medium, respectively.

* Correspondence to: Ghent University, Department of Mathematical Analysis, Research group of Numerical Analysis and Mathematical Modeling (NaM²), Krijgslaan 281 - S8 9000, Ghent, Belgium.

E-mail addresses: kangtong@cuc.edu.cn (T. Kang), karel.vanbockstal@ugent.be (K. Van Bockstal), ranwang@cuc.edu.cn (R. Wang).

URL: http://users.ugent.be/~kavbocks/ (K. Van Bockstal).

https://doi.org/10.1016/j.camwa.2017.10.004

The source term $\boldsymbol{J}_a$ is assumed to be separable, i.e.

$$\boldsymbol{J}_a(\boldsymbol{x},t) = h(t)\boldsymbol{f}(\boldsymbol{x}),$$

where $\boldsymbol{f}(\boldsymbol{x})$ is given but $h(t)$ is unknown. The measurement used to recover $h(t)$ is specified later. It is assumed that the field $\boldsymbol{f}$ obeys

$$\boldsymbol{f} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial\Omega.$$

The whole domain Ω can be divided into finite subdomains. In each subdomain, the electric conductivity σ, the magnetic permeability μ and the electric permittivity ε of the medium are positive constants. However, for simplicity, it is assumed that Ω consists of only two subdomains, namely a conducting subdomain $\Omega_1 =: \Omega_c$ and a nonconducting subdomain $\Omega_e := \Omega_2 = \Omega \setminus \overline{\Omega}_1$. The electric conductivity σ is thus given by

$$\sigma = \begin{cases} \sigma_c & \text{in } \overline{\Omega}_c, \\ 0 & \text{in } \Omega \setminus \overline{\Omega}_c, \end{cases}$$

with $0 < \sigma_c < \infty$. Further, it is assumed that

$$0 < \mu_{\min} \le \mu \le \mu_{\max} \quad \text{and} \quad 0 < \varepsilon_{\min} \le \varepsilon \le \varepsilon_{\max},$$

with $\mu_{\min}$, $\mu_{\max}$, $\varepsilon_{\min}$ and $\varepsilon_{\max}$ all strict positive constants. The interface between both domains is denoted by $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$.

Following the setting formulated above, the system (1.1) is equipped with the following boundary and initial conditions

$$\begin{cases} \boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{0}, & (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \\ \boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0, & \boldsymbol{x} \in \Omega, \\ [\boldsymbol{n}_{12} \cdot (\varepsilon\partial_t \boldsymbol{E} + \sigma\boldsymbol{E} + h\boldsymbol{f})] = [\boldsymbol{n}_{12} \cdot (\nabla \times \boldsymbol{H})] = \boldsymbol{0}, & \text{on } \Gamma \times (0,T], \\ \mu(\boldsymbol{x})\boldsymbol{H}(\boldsymbol{x},t) \cdot \boldsymbol{n} = \boldsymbol{0}, & (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \\ \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0, & \boldsymbol{x} \in \Omega, \\ [\boldsymbol{n}_{12} \times \boldsymbol{H}] = \boldsymbol{0}, & \text{on } \Gamma \times (0,T], \end{cases} \tag{1.2}$$

with

$$(\nabla \cdot \boldsymbol{B}_0)(\boldsymbol{x}) = (\nabla \cdot (\mu\boldsymbol{H}_0))(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega,$$

to ensure that the magnetic induction is divergence free. The unit normal vector $\boldsymbol{n}_{12}$ points from Ω_1 to Ω_2, thus the jump is defined as follows

$$[\boldsymbol{n}_{12} \times \boldsymbol{u}] = \boldsymbol{n} \times (\boldsymbol{u}_2 - \boldsymbol{u}_1), \quad [\boldsymbol{n}_{12} \cdot \boldsymbol{u}] = \boldsymbol{n} \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1).$$

Usually, the Maxwell system is transformed to the $\boldsymbol{E}$ or $\boldsymbol{H}$ equation and the resulting equation is solved by using edge finite element methods. Besides, it can also be changed into potential formulations by means of a decomposition of the field $\boldsymbol{E}$ or $\boldsymbol{H}$ (the so-called $\boldsymbol{A}$-ϕ or $\boldsymbol{T}$-ψ method). Then, nodal finite elements are used to solve the equation numerically, cf. [1–7]. There are several advantages for the potential field method. For example, it can deal with the possible discontinuity between different mediums very well and has good numerical accuracy. The method avoids spurious solutions by adding a penalty function term in the dominant equation. Moreover, it also has attractive features including natural coupling to moment and boundary element methods, and global energy conservation.

There are many works on inverse problems for electromagnetic fields based on the $\boldsymbol{E}$ or $\boldsymbol{H}$ equation, e.g. [8–13]. But research upon the potential field method has been rarely found. The purpose of this paper is to study the inverse source problem specified above by employing the $\boldsymbol{A}$-ϕ method. To transform the formulation (1.1) to the $\boldsymbol{A}$-ϕ method, first a potential vector $\boldsymbol{A}$ is introduced, defined by (see [14, Theorem 2.2])

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \quad \text{with} \quad \nabla \cdot \boldsymbol{A} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{n} \times \boldsymbol{A} = \boldsymbol{0} \text{ on } \partial\Omega,$$

and thus (using the second equation in (1.1))

$$\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\partial_t\phi$$

where the function ϕ is determined up to an additive constant. To ensure the uniqueness of ϕ, it is assumed that $\phi = 0$ on $\partial\Omega$. Now, the first equation of (1.1) can be rewritten as

$$\varepsilon\partial_{tt}(\boldsymbol{A} + \nabla\phi) + \sigma\partial_t\left(\boldsymbol{A} + \nabla\phi\right) + \nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}\right) = h\boldsymbol{f}, \tag{1.3}$$

Moreover, the divergence of both sides of (1.3) has been taken to get that

$$\nabla \cdot \left(\varepsilon\partial_{tt}\left(\boldsymbol{A} + \nabla\phi\right) + \sigma\partial_t\left(\boldsymbol{A} + \nabla\phi\right)\right) = h\nabla \cdot \boldsymbol{f}.$$