Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence and multiplicity of solutions for a nonlocal problem with critical Sobolev exponent[☆]

Jia-Feng Liao [a,b,*], Hong-Ying Li [a], Peng Zhang [b]

[a] School of Mathematics and Information, China West Normal University, Nanchong, Sichuan 637002, People's Republic of China
[b] School of Mathematics and Computational Science, Zunyi Normal College, Zunyi, Guizhou 563002, People's Republic of China

ARTICLE INFO

Article history:
Received 7 June 2017
Received in revised form 13 September 2017
Accepted 15 October 2017
Available online xxxx

Keywords:
Nonlocal problem
Critical exponent
Multiplicity
Variational method
The critical point theorem

ABSTRACT

In this work, we are interested in considering the following nonlocal problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=\mu|u|^{2^*-2}u+\lambda|u|^{q-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N(N\geq 4)$ is a smooth bounded domain, $a\geq 0$, $b>0$, $1<q<2$, $\mu,\lambda>0$ and $2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent. By using the variational method and the critical point theorem, some existence and multiplicity results are obtained.

1. Introduction and main results

In this paper, we consider the following nonlocal problem with critical Sobolev exponent

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=\mu|u|^{2^*-2}u+\lambda|u|^{q-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N(N\geq 3)$ is a smooth bounded domain, $a\geq 0$, $b>0$, $1<q<2$, $\mu,\lambda>0$ are real parameters. $2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for every $p\in[1,2^*]$, where $H_0^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\|=\left(\int_\Omega|\nabla u|^2dx\right)^{\frac{1}{2}}$.

Problem (1.1) was firstly proposed by Kirchhoff in 1883 as a model given by the stationary analogue of equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=0, \tag{1.2}$$

where ρ is the mass density, P_0 is the initial tension, h represents the area of the cross-section, E is the Young modulus of the material and L is the length of the string, see [1]. Thus problem (1.1) is always called the Kirchhoff-type problem. The above

☆ Supported by the Fundamental Research Funds of China West Normal University (No. 16E014), Meritocracy Research Funds of China West Normal University (No. 17YC383), Natural Science Foundation of Education of Guizhou Province (No. KY[2016]046), Science and Technology Foundation of Guizhou Province (No. LH[2016]7033).

* Corresponding author at: School of Mathematics and Information, China West Normal University, Nanchong, Sichuan 637002, People's Republic of China.

E-mail address: liaojiafeng@163.com (J.-F. Liao).

https://doi.org/10.1016/j.camwa.2017.10.012
0898-1221/© 2017 Elsevier Ltd. All rights reserved.

model is an extension of the classical D'Alembert's wave equation by taking into account the changes in the length of the string during the transverse vibrations. It is worth pointing out that problem (1.2) received much attention only after the work of Lions [2] where a function analysis framework was proposed to the problem. After that, the Kirchhoff-type problem has been extensively investigated, such as [3–28].

When $a = 1$, $b = 0$, problem (1.1) reduces the semilinear elliptic equation

$$\begin{cases} -\Delta u = \mu|u|^{2^*-2}u + \lambda|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

In [29], Ambrosetti, Brézis and Cerami studied problem (1.3) with $\mu = 1$ and got some classic existence and multiplicity results by the variational method. In 2012, Figueiredo and Santos researched problem (1.1) with $N = 3$ and $\mu = 1$. By the Krasnoselskii genus, they obtained infinitely many solutions of problem (1.1), see [9]. Later, Sun and Liu [24] considered problem (1.1) with $N = 3$ and $\mu = 1$, and obtained the existence of positive solutions based on the Nehari manifold, Ekelands variational principle and the concentration compactness principle. Naimen got the existence of positive solutions for problem (1.1) with $N = 4$ and $2 \le q < 4$ by the variational methods and concentration compactness argument, see [22]. Very recently, Lei, Liu and Guo [11] also researched problem (1.1) with $N = 3$ and $\mu = 1$. When $b > 0$ small enough, they obtained two positive solutions. In [10], Huang, Liu and Wu studied problem (1.1) with $2 \le q < 2^*$ in a ball. By the Nehari method and some analysis techniques, they got the existence of positive radial solutions and gave a partial answer to Naimen's open problems in [22].

Let S be the best Sobolev constant, namely

$$S := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}. \tag{1.4}$$

Inspired by the works in [10,11,22] and [24], we study the existence and multiplicity of solutions for problem (1.1) with $N \ge 4$. First, we obtain the existence of positive solutions for problem (1.1) with $\mu > 0$ by some analysis techniques and the minimax method. Then, we consider the case $N = 4$, $\mu > bS^2$ and $b > 0$ small enough, obtain the second positive solution by the mountain-pass lemma. Finally, when $\mu > 0$ small, thanks to a global Palais–Smale compactness condition, we get infinitely many pairs of distinct solutions for problem (1.1).

The energy functional corresponding to problem (1.1) is given by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\mu}{2^*}\int_\Omega |u|^{2^*}dx - \frac{\lambda}{q}\int_\Omega |u|^q dx, \quad \forall u \in H_0^1(\Omega).$$

In general, a function u is called a weak solution of problem (1.1) if $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ it holds

$$(a + b\|u\|^2)\int_\Omega (\nabla u, \nabla\varphi)dx - \mu\int_\Omega |u|^{2^*-2}u\varphi dx - \lambda\int_\Omega |u|^{q-2}u\varphi dx = 0.$$

Now our main results can be described as follows:

Theorem 1.1. *Assume $N \ge 4$, $a, b > 0$, $1 < q < 2$ and $\mu > 0$, then there exists $\lambda_* > 0$ such that problem (1.1) has at least one positive solution u_* with $I(u_*) < 0$ for all $0 < \lambda < \lambda_*$.*

Remark 1.1. On the one hand, comparing with [11] and [24], Theorem 1.1 complements their results with $N \ge 4$. On the other hand, Theorem 1.1 supplements the corresponding of the results in [10] and [22] with $1 < q < 2$. Moreover, when $N = 4$, $0 < \mu \le bS^2$ or $N \ge 5$, we can prove that u_* is a positive ground state solution of problem (1.1).

Theorem 1.2. *Assume $N = 4$, $a > 0$, $1 < q < 2$, $\mu > bS^2$ and $b > 0$ small enough, then there exists $\lambda_{**} > 0$ such that problem (1.1) has at least another positive solution u_{**} with $I(u_{**}) > 0$ for all $0 < \lambda < \lambda_{**}$.*

Remark 1.2. Theorem 1.2 replenishes Theorem 2.6 of [11] with $N = 4$.

Let μ^* be a constant by

$$\mu^* = \begin{cases} bS^2, & \text{N} = 4, \\ \left(\frac{2a}{4-2^*}\right)^{\frac{4-2^*}{2}} S^{\frac{2^*}{2}} \left(\frac{2b}{2^*-2}\right)^{\frac{2^*-2}{2}}, & \text{N} \ge 5. \end{cases}$$

We have the following multiplicity result.

Theorem 1.3. *Assume $N \ge 4$, $a, b > 0$, $1 < q < 2$ and $0 < \mu < \mu^*$, then problem (1.1) has infinitely many pairs of distinct solutions for all $\lambda > 0$.*

Please cite this article in press as: J.-F. Liao, et al., Existence and multiplicity of solutions for a nonlocal problem with critical Sobolev exponent, Computers and Mathematics with Applications (2017), https://doi.org/10.1016/j.camwa.2017.10.012.