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Inhomogeneous lossy waveguide mode analysis

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ABSTRACT

This paper discusses electromagnetic numerical mode analysis in waveguides with materially inhomogeneous cross-sections and material dissipation. A full-wave formulation of Maxwell's homogeneous equations including Gauss electric law, stable at vanishing propagation constant is implemented and verified in terms of the *hp*-adaptive version of the finite element method. It provides the possibility to use high order polynomial enrichments combined with strongly graded meshes. It is considered most efficient in resolving the loss of solution regularity at material interfaces with large contrast. Numerical examples including materially lossless homogeneous and inhomogeneous cross sections with and without losses are analysed to corroborate the implementation. The efficiency of using higher order polynomial enrichments is shown. The approach is anticipated to have a broad application, from modern on-chip interconnect and antenna technologies to the design of low observable aerial vehicles.

1. Introduction

The analysis of the propagation of electromagnetic waves in cylinder-like structures whose lengths are much larger than their diameter ($L/D \gg 1$) is of great practical interest in computational electromagnetics for the design of antenna waveguides, on-chip interconnect structures, optical fibres and low radar signature fighter aircraft jet-engine air intakes, to mention a few. A waveguide is here a structure with $L/D \gg 1$ and with at least one open end. It is understood to be (piecewise) materially homogeneous in the axial direction. An ideal lossless waveguide may in this context be thought of as a hollow, air-filled, metallic cylinder where the metallic casing is assumed to be perfectly conducting (PEC). The three dimensional description of the propagating electric and magnetic waves $\{\boldsymbol{E}, \boldsymbol{H}\}(\boldsymbol{x}, t)$, $\boldsymbol{x} : (x, y, z) \in \mathbb{R}^3$ considered in such a waveguide is commonly described as harmonic with respect to the time-variable t and with respect to the axial coordinate z, respectively, $\{\boldsymbol{E}, \boldsymbol{H}\}(\boldsymbol{r})e^{i(\omega t \pm \gamma z)}$, $\boldsymbol{r} \in \mathbb{R}^2$ where ω is the (given) driving circular frequency and γ is the axial propagation parameter, to be determined together with the two-dimensional vector-valued fields $\{\boldsymbol{E}, \boldsymbol{H}\}(\boldsymbol{r})$, see e.g. [1]. Recalling that Maxwell's equations are linear, this most characteristic waveguide feature suggests factoring out the time-dependence and determining for example the two-dimensional electric phasor $\boldsymbol{E}(\boldsymbol{r})$ and common propagation parameter γ by solving the relevant homogeneous form of the Maxwell equations. A forced waveguide response problem may then be solved using a standard mode superposition technique. The partial separation of variables ansatz for a waveguide reduces the spatial dimensionality of the boundary value problem to be solved. Still if the wave-guide cross-section does not have a canonical shape (e.g. rectangular or circular) and its characteristic cross-section diameter to the driving wavelength ratio is large $D/\lambda \gg 1$ the computational problem requires a numerical solution and may become challenging, see Zdunek and Rachowicz [2]. The method preferred here, having lossy materially non-homogeneous cross sections in mind, is the finite

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element method (FEM). Common finite element formulations can be found in many textbooks, see for example Jin [3] and Volakis et al. [4]. A material interface (discontinuity in the electric permittivity and/or the magnetic permeability) within the cross-section implies a loss of regularity in the phasors $\{\boldsymbol{E}, \boldsymbol{H}\}(\boldsymbol{r})$ across the interface, see Costabel, Dauge and Nicase [5]. The regularity loss suggests the use of the *hp*-adaptive version of FEM, see Demkowicz [6]. The materially non-homogeneous case may be solved using a full-wave formulation, i.e. a direct solution of all three components of the chosen electromagnetic phasor. See for example Vardapetyan and Demkowicz [7] or the alternative by Lee, Cendes and Sun [8]. The former is preferred here since it is shown to be stable in the event the propagation parameter becomes small $\gamma \to 0$. In passing it is recalled that the materially homogeneous lossless case posed on a simply connected cross section requires only the solution of two, two-dimensional, scalar eigen problems for the cut-off wave lengths k_c^2 corresponding to the transversely electric and magnetic (TE/TM)-modes respectively. In case the cross-section is multiply connected a third homogeneous problem for the so-called TEM-modes must be solved. Moreover the computed eigen-pairs are independent of the driving frequency ω. In the materially non-homogeneous case, the eigen-solution is ω-dependent. Further, Fernandez et al. [9] showed numerically that the propagation parameter may become complex at certain frequencies ω even in a lossless materially inhomogeneous waveguide, i.e. $\gamma = \alpha + i\beta$. In other words the eigenvalues may be complex although the matrices in the pencil are real-valued. Complex waves always occur in pairs, with the propagation constant of one being the complex conjugate of the other. Since they appear in pairs that do not carry any net power, no dissipation takes place. It has been observed that complex waves have to be included in the field expansion used in field matching procedures for the analysis of waveguide discontinuities and that their omission might lead to serious errors, see Fernandez et al. [9] and references therein.

An *hp*-adaptive implementation of the Vardapetyan and Demkowicz [7] formulation is made. It requires the development of non-standard mixed type edge- and node-based $\mathbf{H}(\text{curl})$- and H^1-conforming elements. Material losses are included; the electric permittivity and/or the magnetic permeability may be complex-valued. An electric field based formulation is chosen. Given the driving frequency ω we compute the complex-valued propagation parameter γ and the corresponding two-dimensional tangential and the scalar axial part of the eigen-field $\mathbf{e} = (\mathbf{e}_t, \mathrm{e}_z)$ directly. The resulting algebraic eigen-problem is of the generalised type, $(\mathbf{A} - \gamma^2\mathbf{B})\boldsymbol{e} = \mathbf{0}$. It has a complex-valued pencil $(\mathbf{A}, \mathbf{B})$ in presence of dissipative material contributions. Matrix $\mathbf{A}(\omega)$ is ω-dependent and non-symmetric while matrix $\mathbf{B}$ is constant, block-diagonal with a zero diagonal block. The problem is solved with a modularised shift-invert technique. The numerical algorithm developed consists of a MPI/OpenMP parallelised so-called multi-frontal solver for non-symmetric complex-valued unassembled matrices with input domain-wise element-by-element for which a domain-decomposition is created providing a good load-balancing. Further an Arnoldi-based eigen solver for complex-valued problems is employed in a shift-invert mode. The algorithm is implemented in the *hp*-adaptive code *hp2d* provided with the textbook by Demkowicz [6]. The use of high order *p*-enrichments is exercised and shown to be very efficient.

The remainder of this article is disposed as follows. The materially inhomogeneous waveguide problem is stated and formulated in Section 2. The associated algebraic problem is set up and the solution algorithm used is presented in Section 3. The numerical examples used to verify the implementation and to illustrate the applicability are found in Section 4. We consider a simple rectangular inhomogeneous waveguide benchmark problem in Section 4.1. We provide our numerical results for the shielded dielectric image line investigated thoroughly by Strube and Arndt [10] in Section 4.2. Finally in Section 4.3 we illustrate the travelling wave analysis of a circular wave guide with and without a lossy coating. The article is concluded by a short discussion in Section 5 and a summary and conclusions in Section 6.

2. Problem statement

Given a driving circular frequency, geometry and boundary data for a wave-guide, including material data for a possibly non-homogeneous cross section, we seek the propagation constants and the associated out-of-plane wave modes. To this end the following time harmonic condition for the electric field intensity,

$$\tilde{\boldsymbol{E}}(\boldsymbol{x}, t) = \mathfrak{Re}\left[\boldsymbol{E}(\boldsymbol{x})e^{i\omega t}\right], \tag{1}$$

is assumed with $\boldsymbol{x} \in \mathbb{R}^3$, where ω is the circular frequency and $i = \sqrt{-1}$ is the imaginary unit and $\boldsymbol{E}(\boldsymbol{x})$ is the complex-valued phasor field. Let $\mathbf{H}(\text{curl}, \Omega)$ be the set of square integrable vector fields on a domain $\Omega \subset \mathbb{R}^3$ whose curl is square integrable, and let $H^1(\Omega)$ be the set of square integrable scalar fields on the domain Ω whose first partial derivatives are square integrable. The full-wave, three-dimensional time harmonic Maxwell eigenvalue-problem can be stated in weak form as:

$$\begin{cases} \text{Find} \quad \boldsymbol{E} \in \mathbf{H}(\text{curl}, \Omega), \quad \boldsymbol{n} \times \boldsymbol{E} = \mathbf{0} \quad \text{on} \quad \partial\Omega_{\text{PEC}}, \quad \text{and} \quad \gamma^2 \in \mathbb{C}: \\ \left(\mu_r^{-1}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{F}\right) - \gamma^2 (\epsilon_r \boldsymbol{E}, \boldsymbol{F}) = \mathbf{0}, \qquad \forall \boldsymbol{F} \in \mathbf{H}(\text{curl}, \Omega), \quad \boldsymbol{n} \times \boldsymbol{F} = \mathbf{0} \quad \text{on} \quad \partial\Omega_{\text{PEC}} \\ (\epsilon_r \boldsymbol{E}, \nabla\varphi) = 0, \qquad \forall\varphi \in H^1(\Omega), \quad \varphi = 0 \quad \text{on} \quad \partial\Omega_{\text{PEC}}, \end{cases} \tag{2a,b}$$

where as usually μ_r is the relative magnetic permeability and ϵ_r is the relative electric permittivity, and where we use the standard $L^2(\Omega)$ inner product notation,

$$(\boldsymbol{A}, \boldsymbol{B})_\Omega = \int_\Omega \boldsymbol{A} \cdot \bar{\boldsymbol{B}} \, d\Omega. \tag{3}$$

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