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A variant of the HSS preconditioner for complex symmetric indefinite linear systems

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ABSTRACT

Using the equivalent block two-by-two real linear systems, we establish a new variant of the Hermitian and skew-Hermitian splitting (HSS) preconditioner for a class of complex symmetric indefinite linear systems. The new preconditioner is not only a better approximation to the block two-by-two real coefficient matrix than the well-known HSS preconditioner, but also resulting in an unconditional convergent fixed-point iteration. The quasi-optimal parameter, which minimizes an upper bound of the spectral radius of the iteration matrix, is analyzed. Eigen-properties and an upper bound of the degree of the minimal polynomial of the preconditioned matrix are discussed. Finally, two numerical examples are provided to show the efficiency of the new preconditioner.

1. Introduction

Consider the iterative solution of the large sparse system of linear equations

$$Az = b, \quad \text{with} \quad A = W + \mathrm{i}T, \quad z = x - \mathrm{i}y \quad \text{and} \quad b = g + \mathrm{i}f, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a complex symmetric matrix, $W, T \in \mathbb{R}^{n\times n}$ are real symmetric matrices, the vectors x, y, f, g are all in $\mathbb{R}^n$, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. We further assume that $null(W) \bigcap null(T) = \{0\}$ and $\mathrm{i} = \sqrt{-1}$ is not a generalized eigenvalue of the matrix pair (W, T), where $null(\cdot)$ denotes the null space of the corresponding matrix. Based on these assumptions, we know that the complex symmetric matrix A is nonsingular and the linear system (1.1) has a unique solution.

Complex linear systems of this kind are very important and arise from many fields of study, such as the frequency analysis of linear mechanical systems [1], Helmholtz equations [2,3], optical tomography problem [4], eddy current problems [5,6], electrical power system modeling [7], quantum mechanics [8] and so on. For more application backgrounds, we refer the readers to [9–12] and references therein for details.

In recent years, many authors have contributed to the development of the efficient iteration methods for solving the complex symmetric linear system (1.1). In general, there are two approaches. The first one is to tackle the $n \times n$ complex symmetric linear system (1.1) directly, such as the conjugate orthogonal conjugate gradient method [13], the quasi-minimal residual iteration method [14], the Hermitian and skew-Hermitian splitting (HSS) iteration method [15,16] and so on. Due to its promising performance and elegant mathematical properties, the HSS iteration method has attracted many researchers' attention. It is easy to see that W and $\mathrm{i}T$ are the Hermitian part and the skew-Hermitian part of the complex symmetric

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matrix A, respectively. Let α be a real positive parameter and I be the identity matrix. Then the HSS iteration scheme is defined as

$$\begin{cases}(\alpha I+W)z^{(k+\frac{1}{2})}=(\alpha I-\mathrm{i}T)z^{(k)}+b,\\(\alpha I+\mathrm{i}T)z^{(k+1)}=(\alpha I-W)z^{(k+\frac{1}{2})}+b.\end{cases}\quad(k=0,1,2,\ldots) \tag{1.2}$$

To avoid solving the linear system with the complex coefficient matrix $\alpha I+\mathrm{i}T$, Bai, Benzi and Chen skillfully designed a modified Hermitian and skew-Hermitian splitting (MHSS) method [10] to solve the complex symmetric linear system (1.1)

$$\begin{cases}(\alpha I+W)z^{(k+\frac{1}{2})}=(\alpha I-\mathrm{i}T)z^{(k)}+b,\\(\alpha I+T)z^{(k+1)}=(\alpha I+\mathrm{i}W)z^{(k+\frac{1}{2})}-\mathrm{i}b.\end{cases}\quad(k=0,1,2,\ldots) \tag{1.3}$$

It has been studied in [10,15] that if W is positive definite and T is positive semidefinite, then both the HSS iteration method (1.2) and the MHSS iteration method (1.3) are convergent unconditionally. The corresponding optimal parameter is $\alpha^*=\sqrt{\lambda_{max}(W)\lambda_{min}(W)}$, which minimizes the upper bounds of the iteration matrices of the HSS iteration matrix and the MHSS iteration matrix. To accelerate the convergence rate of the MHSS iteration method, many works have been done [17,18]. For example, by preconditioning the complex symmetric linear system first and then iterating with the MHSS iteration method, the preconditioned MHSS iteration method was proposed in [19]. By extrapolating the MHSS iteration sequence with two different parameters, the generalized PMHSS iteration method was studied in [20]. By using the lopsided technique studied for solving general non-Hermitian positive definite linear systems, the lopsided MHSS iteration method was studied in [21]. However, all these works are studied when the matrix W is symmetric positive definite. For a broad class of problems, the matrix W is symmetric indefinite and the above iteration methods may be divergence. When W and T are symmetric indefinite matrices and satisfy $-W\preceq T\prec W$ or $-T\preceq W\prec T$, Xu [2] presented a generalization of the PMHSS iteration method. Here and in the sequel, for any matrices B and C, $B\prec(\preceq)C$ means that $C-B$ is symmetric positive (semi)definite. Recently, Cao and Ren [22] proposed two variants of the PMHSS iteration methods and proved the convergence. Wu [23] designed the Hermitian and normal splitting (HNS) iteration method. However, the application of these methods to the complex system (1.1) requires complex arithmetic which one may wish to avoid.

Another approach is to deal with the equivalent block two-by-two real linear systems

$$\hat{\mathcal{A}}\hat{u}=\begin{bmatrix}W&-T\\T&W\end{bmatrix}\begin{bmatrix}x\\-y\end{bmatrix}=\begin{bmatrix}g\\f\end{bmatrix}=\hat{b}, \tag{1.4}$$

or

$$\mathcal{A}u=\begin{bmatrix}T&-W\\W&T\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}f\\g\end{bmatrix}=b, \tag{1.5}$$

which can be solved in real arithmetics by the HSS iteration method or some Krylov subspace iteration methods (such as GMRES). In this paper, we focus on the complex symmetric indefinite linear system (1.1) where W is symmetric indefinite and T is symmetric positive definite. Thus, the coefficient matrix $\hat{\mathcal{A}}$ and $\mathcal{A}$ are nonsymmetric indefinite and nonsymmetric positive definite (i.e., its symmetric part is positive definite), respectively. For such case, many iteration methods, such as the MHSS iteration method, the lopsided MHSS iteration method and their preconditioned variants, are divergent. The following HSS iteration scheme

$$\begin{cases}\begin{bmatrix}\alpha I+T&0\\0&\alpha I+T\end{bmatrix}\begin{bmatrix}x^{(k+\frac{1}{2})}\\y^{(k+\frac{1}{2})}\end{bmatrix}=\begin{bmatrix}\alpha I&W\\-W&\alpha I\end{bmatrix}\begin{bmatrix}x^{(k)}\\y^{(k)}\end{bmatrix}+\begin{bmatrix}f\\g\end{bmatrix},\\\begin{bmatrix}\alpha I&-W\\W&\alpha I\end{bmatrix}\begin{bmatrix}x^{(k+1)}\\y^{(k+1)}\end{bmatrix}=\begin{bmatrix}\alpha I-T&0\\0&\alpha I-T\end{bmatrix}\begin{bmatrix}x^{(k+\frac{1}{2})}\\y^{(k+\frac{1}{2})}\end{bmatrix}+\begin{bmatrix}f\\g\end{bmatrix},\end{cases}\quad(k=0,1,2,\ldots) \tag{1.6}$$

may be suitable for solving the equivalent block two-by-two real linear system (1.5). In general, any matrix splitting not only can automatically lead to a splitting iteration method, but also can naturally induce a splitting preconditioner for accelerating the convergence rate of the Krylov subspace methods. For the block two-by-two real linear system (1.5), the HSS preconditioner is

$$\mathcal{P}_{HSS}=\frac{1}{2\alpha}\begin{bmatrix}\alpha I+T&0\\0&\alpha I+T\end{bmatrix}\begin{bmatrix}\alpha I&-W\\W&\alpha I\end{bmatrix}. \tag{1.7}$$

It has been studied in [15,24] that the HSS iteration method (1.6) is convergent unconditionally to solve the block two-by-two real linear system (1.5) and all eigenvalues of the HSS preconditioned matrix $\mathcal{P}_{HSS}^{-1}\mathcal{A}$ are located in a circle centered at (1,0) with radius strictly less than one. However, the convergence rate of the HSS preconditioned iteration method is slow. Recently, based on the relaxed techniques studied in [25–27], Zhang and Dai presented a block splitting (BS) preconditioner

$$\mathcal{P}_{BS}=\begin{bmatrix}I&-W\\\frac{1}{\alpha}W&T\end{bmatrix}\begin{bmatrix}\alpha I+T&0\\0&I\end{bmatrix}=\begin{bmatrix}\alpha I+T&-W\\W(I+\frac{1}{\alpha}T)&T\end{bmatrix}. \tag{1.8}$$