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Multilevel sparse grids collocation for linear partial differential equations, with tensor product smooth basis functions

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ABSTRACT

Radial basis functions have become a popular tool for approximation and solution of partial differential equations (PDEs). The recently proposed multilevel sparse interpolation with kernels (MuSIK) algorithm proposed in Georgoulis et al. (2013) shows good convergence. In this paper we use a sparse kernel basis for the solution of PDEs by collocation. We will use the form of approximation proposed and developed by Kansa (1986). We will give numerical examples using a tensor product basis with the multiquadric (MQ) and Gaussian basis functions. This paper is novel in that we consider space–time PDEs in four dimensions using an easy-to-implement algorithm, with smooth approximations. The accuracy observed numerically is as good, with respect to the number of data points used, as other methods in the literature; see Langer (2016) and Wang et al. (2016).

1. Introduction

During the last few decades since radial basis functions (RBFs) were proposed by Hardy [1] for numerical approximation, they have been applied to a wide range of applications from mathematics, geophysics, physics to engineering and finance. In this paper we will use tensor products of the infinitely differentiable univariate functions

$$\text{Multiquadric}\ :\ \phi_c(x) = \sqrt{x^2 + c^2},$$
$$\text{Gaussian}\ :\ \psi_c(x) = e^{-\frac{x^2}{c^2}}.$$

The basis function for approximation is then of the form

$$\Phi_{\mathbf{c}}(\mathbf{x}) = \prod_{i=1}^{d} \mu_{c_i}(x_i),$$

where μ is either ϕ or ψ. This is not strictly speaking RBF approximation in general, though for the Gaussian basis function, since

$$\prod_{i=1}^{d} \exp\left(-x_i^2\right) = \exp\left(-\left(\sum_{i=1}^{d} x_i^2\right)\right) = \exp\left(-\|\mathbf{x}\|^2\right),$$

we obtain a univariate function of the norm (an RBF).

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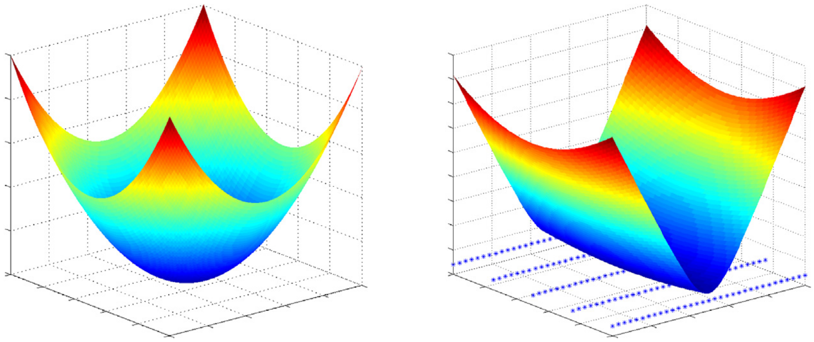

Fig. 1. An example of normal MQ and anisotropic tensor MQ functions in 2D. The anisotropic function to the right is scaled appropriately for the anisotropic grid shown.

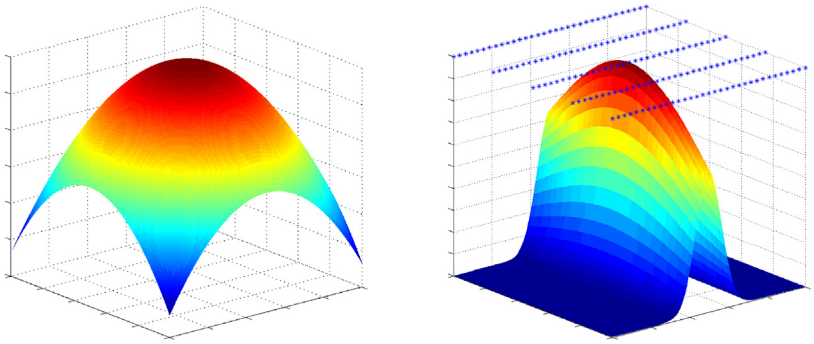

Fig. 2. An example of normal Gaussian and anisotropic tensor Gaussian functions in 2D. The anisotropic function to the right is scaled appropriately for the anisotropic grid shown.

In the definition of the multiquadric and Gaussian there is a parameter c which we call the shape parameter. This is used to scale the approximation basis in various directions depending on the resolution of the data points in that direction; see Figs. 1 and 2. In the first we plot $\Phi_{[1,1]}(\mathbf{x}) = \phi_1(x_1)\phi_1(x_2)$ and $\Phi_{[1/2,1/32]}(\mathbf{x}) = \phi_{1/2}(x_1)\phi_{1/32}(x_2)$, and in the second $\Phi_{[1,1]}(\mathbf{x}) = \psi_1(x_1)\psi_1(x_2)$ and $\Phi_{[1/2,1/32]}(\mathbf{x}) = \psi_{1/2}(x_1)\psi_{1/32}(x_2)$. The scaling matches the anisotropic grid shown next to the surface. We call the basis functions with different shape in each direction anisotropic basis functions.

More recently RBFs have been employed in the solution of PDEs [2–12]. Suppose our PDE is

$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega. \tag{2}$$

There are two distinct collocations methods using RBFs in this context, termed symmetric and non-symmetric collocation. The latter was introduced by Kansa [13–15] and involves the expansion of the solution of the PDE in a combination of RBFs:

$$\tilde{u}(\mathbf{y}) = \sum_i \alpha_i \Phi_{\mathbf{c}_i}(\mathbf{y} - \mathbf{x}_i),$$

where the nodes $\mathbf{x}_i \in \Omega \cup \partial\Omega$. The PDE is applied to this expansion and collocation is used to compute coefficients in the expansion. Currently there is no proof that this method is stable in the sense that the collocation system is invertible. However, the method remains simple to implement and shows good convergence. In symmetric collocation, developed by [16], the solution of the PDE is written in the form:

$$\tilde{u}(\mathbf{y}) = \sum_i \alpha_i \mathcal{L}\Phi_{\mathbf{c}_i}(\mathbf{y} - \mathbf{x}_i) + \sum_j \beta_j \Phi_{\mathbf{c}_j}\left(\mathbf{y} - \mathbf{z}_j\right),$$

where now $\mathbf{x}_i \in \Omega$, and $\mathbf{z}_j \in \partial\Omega$. Now the collocation system which arises is symmetric and for specific choices of positive definite RBFs (the Gaussian for instance) the system can be proven to be invertible (see [16]).

Due to the simplicity of implementation we will use non-symmetric collocation in this paper. We will explore the use of symmetric collocation with sparse grids in a follow-up article.

One of the advantages in using radial basis functions is the ease of implementation in high dimensional problems, though this is of no practical consequence if we cannot mitigate the so-called *curse of dimensionality*. The sparse grid methodology

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