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Two-grid method for two-dimensional nonlinear Schrödinger equation by mixed finite element method ☆

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ABSTRACT

A conservative two-grid mixed finite element scheme is presented for two-dimensional nonlinear Schrödinger equation. One Newton iteration is applied on the fine grid to linearize the fully discrete problem using the coarse-grid solution as the initial guess. Moreover, error estimates are conducted for the two-grid method. It is shown that the coarse space can be extremely coarse, with no loss in the order of accuracy, and still achieve the asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{2}})$ in the two-grid method. The numerical results show that this method is very effective.

1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most important equations of mathematical physics with applications in many fields, such as plasma physics, nonlinear optics, water waves, bimolecular dynamics, and protein chemistry. In this paper, we consider the following initial–boundary value problem of NLS equation in two dimension:

$$i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = 0, \quad (x, y, t) \in \Omega \times (0, T], \tag{1.1}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \bar{\Omega}, \tag{1.2}$$

$$u(x, y, t) = 0, \quad (x, y) \in \Gamma, \quad t \in J = (0, T], \tag{1.3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, Ω is $[x_l, x_r] \times [y_l, y_r]$, Γ is the boundary of Ω, $u(x, y, t)$ is a complex function, $u_0(x, y)$ is prescribed smooth complex function, and $i^2 = -1$.

Computing the inner product of Eq. (1.1) with u and $\frac{\partial u}{\partial t}$, and then taking the imaginary part and the real part, respectively, the two conservative laws are obtained as follows

$$Q(t) = \|u(\cdot, t)\| = \|u_0\| = Q(0), \tag{1.4}$$

$$E(t) = \|u(\cdot, t)\|_{H^1}^2 + \frac{1}{2}\|u(\cdot, t)\|_{L^4}^4 = E(0). \tag{1.5}$$

Zhang et al. found that nonconservative schemes are subject to nonlinear blow-up when studying the NLS equation, so they presented a conservative difference scheme in [1]. Furthermore, extensive mathematical and numerical studies have been conducted for the NLS equation [2–9].

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In recent years, there has been tremendous interest in developing finite difference schemes and finite element methods for two-dimensional Schrödinger equations [2,10–16]. Wang studied the split-step finite difference method for various cases of NLS equations, such as cubic NLS equations, coupled NLS equations with constant coefficients and Gross–Pitaevskii equations in 1D, 2D, and 3D in [17]. Dehghan presented a compact split-step finite difference method for solving NLS equations with constant and variable coefficients in [18]. [12,15] introduced the alternating direction implicit method for two-dimensional NLS equations, which has the advantage of not increasing the dimensions of the coefficient matrices corresponding to the matrix equations. The multigrid method is an efficient algorithm to solve the NLS equation [19–21].

As we know, Xu [22,23] introduced the two-grid finite element approach to efficiently solve nonlinear elliptic equations, where the basic idea is to use a coarse space to produce a rough approximation of the solution, and then to use the rough approximately as the initial guess for one Newton iteration on the fine grid. This procedure involves a nonlinear solver on the coarse space and a linear solver on the fine space. It is a simple but effective algorithm that has been applied to many types of problems, Dawson [24] and Chen [25,26] proposed a two-grid method for quasilinear reaction diffusion equations. Jin et al. [16] successfully extended the two-grid finite element method to solve coupled partial differential equations, such as the linear Schrödinger equation, where the equations on fine grid are decoupled, so that the computational complexity of solving the Schrödinger equation is comparable to solving two decoupled Poisson equations on the same fine grid. Chien et al. [27] proposed two-grid discretization schemes with two-loop continuation algorithms for nonlinear Schrödinger equations, where the centered difference approximations, the six-node triangular elements and the Adini elements are used to discretize the partial differential equations. Numerical experiments have shown that these schemes are efficient, but no rigorous error analysis was presented. Wu [28,29] constructed two-grid mixed finite element schemes for nonlinear Schrödinger equations, where a linear and indefinite (the typical nature of mixed finite element) discretization systems are solved on the fine grid. Numerical experiments using these schemes have been shown to be efficient; however, no rigorous error analysis has been conducted. Recently, Zhang et al. [30] extended the approach given in [16] to time-dependent linear Schrödinger equation by finite element method. In this paper, we use the mixed finite element method for the Schrödinger equations. We first estimate the finite element solution in the sense of $L^\infty(J; L^p)$ norm. Then, we present our main algorithm—a new two-grid method. There are lots of literatures concerning about the Schrödinger equations using different treatments, but, to the best of our knowledge, there are few results on two-grid algorithms used for such nonlinear coupled problems. The main goal of our algorithm is to solve a nonlinear coupled system in the very coarse grid and then solve the decoupled linear system on the fine grid. It is shown that the coarse grid can achieve the asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{2}})$ in the two-grid algorithm. Under reasonable assumptions, the two-grid algorithm can retain the same order of approximation accuracy as the coupled and nonlinear algorithms and with a much lower time cost.

The remainder of this paper is organized as follows. We present the weak formulas of our model, in Section 2. In Section 3, a simple analysis of the finite element solutions of the model are given. Our main algorithm and its convergence analysis are presented in Section 4. In Section 5, an iterative algorithm and numerical results are discussed.

2. Weak formulation of the problem and preliminaries

Now, we state some standard notations which will be used in this article. We denote $W^{m,p}(\Omega)$ the complex Sobolev space on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p} = \sum_{|\alpha|\le m}\|D^\alpha\phi\|^p_{L^p}$. We set $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$ and $\|\cdot\| = \|\cdot\|_{0,2}$, and the subspace of $H^1(\Omega)$ consisting of functions with vanishing trace on $\partial\Omega$ by $H^1_0(\Omega)$.

Let $(\cdot,\cdot)$ be the inner product on $L^2(\Omega) = H^0(\Omega)$ defined by

$$(u, v) = \int_\Omega u(x)\bar{v}(x)dx, \quad \textit{for } u, v \in L^2(\Omega),$$

and denote by $\|\cdot\|$ the associated L^2 norm.

For any complex-valued function $v = v_1 + iv_2$, let

$$\|v\|_{m,p} = (\|v_1\|^p_{m,p} + \|v_2\|^p_{m,p})^{\frac{1}{p}},$$
$$\|v\|_{m,\infty} = \|v_1\|_{m,\infty} + \|v_2\|_{m,\infty}.$$

Let $H(div;\Omega)$ be the set of vector functions $\boldsymbol{v} \in (L^2(\Omega))^2$, such that $\nabla\cdot\boldsymbol{v} \in L^2(\Omega)$ and let

$$\boldsymbol{V} = H(div;\Omega) \cap \{\boldsymbol{v}\cdot\nu = 0 \text{ on } \partial\Omega\},$$
$$W = \{\varphi \in L^2(\Omega)\},$$

equipped with the norm given by:

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} = \|\boldsymbol{v}\|_{H(div;\Omega)} = (\|\boldsymbol{v}\|^2 + \|\nabla\cdot\boldsymbol{v}\|^2)^{\frac{1}{2}},$$

and $\|\varphi\|$. It is clear $\nabla\cdot\boldsymbol{V} \subseteq W$.

Take $\boldsymbol{z} = \nabla u$. The weak form of (1.1)–(1.3) is equivalent to the problem of finding a map $(u, \boldsymbol{z}) : J \to L^2(\Omega)\times\boldsymbol{V}$ such that

$$i(\frac{\partial u}{\partial t}, \varphi) - (\nabla\cdot\boldsymbol{z}, \varphi) + (|u|^2u, \varphi) = 0, \quad \forall\varphi \in L^2(\Omega), \tag{2.1}$$

$$(\boldsymbol{z}, \chi) + (u, \nabla\cdot\chi) = 0, \quad \forall\chi \in \boldsymbol{V}. \tag{2.2}$$

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