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# Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions

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ABSTRACT

We derive $\boldsymbol{H}(\mathrm{curl})$-error estimates and improved $\boldsymbol{L}^2$-error estimates for the Maxwell equations approximated using edge finite elements. These estimates only invoke the expected regularity pickup of the exact solution in the scale of the Sobolev spaces, which is typically lower than $\frac{1}{2}$ and can be arbitrarily close to 0 when the material properties are heterogeneous. The key tools for the analysis are commuting quasi-interpolation operators in $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming finite element spaces and, most crucially, newly-devised quasi-interpolation operators delivering optimal estimates on the decay rate of the best-approximation error for functions with Sobolev smoothness index arbitrarily close to 0. The proposed analysis entirely bypasses the technique known in the literature as the discrete compactness argument.



## 1. Introduction

The objective of this paper is to review some recent results concerning the approximation of the Maxwell equations using edge finite elements. One important difficulty is the modest regularity pickup of the exact solution in the scale of the Sobolev spaces which is typically lower than $\frac{1}{2}$ and can be arbitrarily close to 0 when the material properties are heterogeneous. We show that the difficulties induced by the lack of stability of the canonical interpolation operators in $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming finite element spaces can be overcome by invoking recent results on commuting quasi-interpolation operators and newly devised quasi-interpolation operators that deliver optimal estimates on the decay rate of the best-approximation error in those spaces. In addition to a curl-preserving lifting operator introduced by Monk [1, p. 249–250], the commuting quasi-interpolation operators are central to establish a discrete counterpart of the Poincaré–Steklov inequality (bounding the $\boldsymbol{L}^2$-norm of a divergence-free field by the $\boldsymbol{L}^2$-norm of its curl), as already shown in the pioneering work of Arnold et al. [2, §9.1] on Finite Element Exterior Calculus. It is therefore possible to bypass entirely the technique known in the literature as the discrete compactness argument (Kikuchi [3], Monk and Demkowicz [4], Caorsi et al. [5]). The novelty here is the use of quasi-interpolation operators devised by the authors in [6] that give optimal decay rates of the approximation error in fractional Sobolev spaces with a smoothness index that can be arbitrarily small. This allows us to establish optimal $\boldsymbol{H}(\mathrm{curl})$-norm and $\boldsymbol{L}^2$-norm error estimates that do not invoke additional regularity assumptions on the exact solution other than those resulting from the model problem at hand. Optimality is understood here in the sense of the decay rates with respect to the mesh-size; the constants in the error estimates can depend on the heterogeneity ratio of the material properties. Note that all the above quasi-interpolation operators are available with or without prescription of essential boundary conditions.

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The paper is organized as follows. Notation and technical results are given in Section 2. The main results from this section are Theorem 2.2, which states the existence of optimal commuting quasi-interpolation operators, and Theorems 2.3 and 2.4, which give decay estimates of the best approximation in fractional Sobolev norms. Section 3 is concerned with standard facts about the Maxwell equations. In particular, we state our main assumptions on the model problem and briefly recall standard approximation results for the Maxwell equations that solely rely on a coercivity argument. The new results announced above are collected in Section 4 and in Section 5. After establishing the discrete Poincaré–Steklov inequality in Theorem 4.5, our main results are Theorem 4.8 for the $\boldsymbol{H}(\text{curl})$-error estimate and Theorem 5.3 for the improved $\boldsymbol{L}^2$-error estimate. Both results do not invoke regularity assumptions on the exact solution other than those resulting from the model problem at hand.

## 2. Preliminaries

We recall in this section some notions of functional analysis and approximation using finite elements that will be invoked in the paper. The space dimension is 3 in the entire paper ($d = 3$) and $D$ is an open, bounded, and connected Lipschitz subset in $\mathbb{R}^3$.

### 2.1. Functional spaces

We are going to make use of the standard $L^2$-based Sobolev spaces $H^m(D)$, $m \in \mathbb{N}$. The vector-valued counterpart of $H^m(D)$ is denoted $\boldsymbol{H}^m(D)$. We additionally introduce the vector-valued spaces

$$\boldsymbol{H}(\text{curl}; D) := \{\boldsymbol{b} \in \boldsymbol{L}^2(D) \mid \nabla \times \boldsymbol{b} \in \boldsymbol{L}^2(D)\}, \tag{2.1}$$

$$\boldsymbol{H}(\text{div}; D) := \{\boldsymbol{b} \in \boldsymbol{L}^2(D) \mid \nabla \cdot \boldsymbol{b} \in L^2(D)\}. \tag{2.2}$$

To be dimensionally coherent, we equip these Hilbert spaces with the norms

$$\|b\|_{H^1(D)} := (\|b\|^2_{L^2(D)} + \ell_D^2 \|\nabla b\|^2_{L^2(D)})^{\frac{1}{2}}, \tag{2.3}$$

$$\|\boldsymbol{b}\|_{\boldsymbol{H}(\text{curl};D)} := (\|\boldsymbol{b}\|^2_{L^2(D)} + \ell_D^2 \|\nabla \times \boldsymbol{b}\|^2_{L^2(D)})^{\frac{1}{2}}, \tag{2.4}$$

$$\|\boldsymbol{b}\|_{\boldsymbol{H}(\text{div};D)} := (\|\boldsymbol{b}\|^2_{L^2(D)} + \ell_D^2 \|\nabla \cdot \boldsymbol{b}\|^2_{L^2(D)})^{\frac{1}{2}}, \tag{2.5}$$

where $\ell_D$ is some characteristic dimension of $D$, say the diameter of $D$ for instance. In this paper we are also going to use fractional Sobolev norms with smoothness index $s \in (0, 1)$, defined as follows:

$$\|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} := (\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^{2s} |\boldsymbol{b}|^2_{\boldsymbol{H}^s(D)})^{\frac{1}{2}}, \tag{2.6}$$

where $|\cdot|_{\boldsymbol{H}^s(D)}$ is the Sobolev–Slobodeckij semi-norm applied componentwise. Similarly, for any $s > 0$, $s \in \mathbb{R} \setminus \mathbb{N}$, and $p \in [1, \infty)$, the norm of the Sobolev space $W^{s,p}(D)$ is defined by $\|v\|_{W^{s,p}(D)} := (\|v\|^p_{W^{m,p}(D)} + \ell_D^{sp} \sum_{|\alpha|=m} |\partial^\alpha v|^p_{W^{\sigma,p}(D)})^{\frac{1}{p}}$ with $\|v\|_{W^{m,p}(D)} := (\sum_{|\alpha| \le m} \ell_D^{|\alpha|p} \|\partial^\alpha v\|^p_{L^p(D)})^{\frac{1}{p}}$ where $m := \lfloor s \rfloor \in \mathbb{N}$, $\sigma := m - s \in (0, 1)$.

### 2.2. Traces

In order to make sense of the boundary conditions, we introduce trace operators. Let $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ be the (full) trace operator. It is known that $\gamma^{\mathrm{g}}$ is surjective. Let $\langle \cdot, \cdot \rangle_{\partial D}$ denote the duality pairing between $\boldsymbol{H}^{-\frac{1}{2}}(\partial D) := (\boldsymbol{H}^{\frac{1}{2}}(\partial D))'$ and $\boldsymbol{H}^{\frac{1}{2}}(\partial D)$. We define the tangential trace operator $\gamma^{\mathrm{c}} : \boldsymbol{H}(\text{curl}; D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ as follows:

$$\langle \gamma^{\mathrm{c}}(\boldsymbol{v}), \boldsymbol{l} \rangle_{\partial D} := \int_D \boldsymbol{v} \cdot \nabla \times \boldsymbol{w}(\boldsymbol{l}) \, \mathrm{d}x - \int_D (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{w}(\boldsymbol{l}) \, \mathrm{d}x, \tag{2.7}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\text{curl}; D)$, all $\boldsymbol{l} \in \boldsymbol{H}^{\frac{1}{2}}(\partial D)$ and all $\boldsymbol{w}(\boldsymbol{l}) \in \boldsymbol{H}^1(D)$ such that $\gamma^{\mathrm{g}}(\boldsymbol{w}(\boldsymbol{l})) = \boldsymbol{l}$. One readily verifies that the definition (2.7) is independent of the choice of $\boldsymbol{w}(\boldsymbol{l})$, that $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D} \times \boldsymbol{n}$ when $\boldsymbol{v}$ is smooth, and that the map $\gamma^{\mathrm{c}}$ is bounded.

We define similarly the normal trace map $\gamma^{\mathrm{d}} : \boldsymbol{H}(\text{div}; D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$\langle \gamma^{\mathrm{d}}(\boldsymbol{v}), l \rangle_{\partial D} := \int_D \boldsymbol{v} \cdot \nabla q(l) \, \mathrm{d}x + \int_D (\nabla \cdot \boldsymbol{v}) q(l) \, \mathrm{d}x, \tag{2.8}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\text{div}; D)$, all $l \in H^{\frac{1}{2}}(\partial D)$, and all $q(l) \in H^1(D)$ such that $\gamma^{\mathrm{g}}(q(l)) = l$. Here $\langle \cdot, \cdot \rangle_{\partial D}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$. One can verify that the definition (2.8) is independent of the choice of $q(l)$, that $\gamma^{\mathrm{d}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D} \cdot \boldsymbol{n}$ when $\boldsymbol{v}$ is smooth, and that the map $\gamma^{\mathrm{d}}$ is bounded.

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