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Analytical and numerical solutions of the unsteady 2D flow of MHD fractional Maxwell fluid induced by variable pressure gradient

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ABSTRACT

The nonlocal property of the fractional derivative can supply more precise mathematical models for depicting flow dynamics of complex fluid which cannot be modelled appropriately by normal integer order differential equations. This paper studies the analytical and numerical methods of unsteady 2D flow of Magnetohydrodynamic (MHD) fractional Maxwell fluid in a rectangular pipe driven by variable pressure gradient. The governing equation is formulated with Caputo time dependent fractional derivatives whose orders are distributed in interval (0, 2). A challenge is to firstly obtain the exact solution by combining modified separation of variables method with Mikusiński-type operational calculus. Meanwhile, the numerical solution is also obtained by the implicit finite difference method whose validity has been confirmed by the comparison with the exact solution constructed. Different to the most classical works, both the stability and convergence analysis of two-dimensional multi-term time fractional momentum equation are derived. Based on numerical analysis, the results show that the velocity increases with the rise of the fractional parameter and relaxation time. While an increase in the values of Hartmann number leads to a slower velocity in the rectangular pipe.

1. Introduction

The researches on rheological equations of viscoelastic fluid with fractional order derivatives have been paid more and more attentions because of its extensive application in industrial and technical fields. Meanwhile, in consideration of the time memory character of fractional order derivative, it is suitably applied to describe constitutive relationship of viscoelastic fluid [1–9]. The fractional constitutive equations of viscoelastic fluids which are generally established from the well-known fluid models by replacing the time derivative of an integer order with the fractional calculus operators. Many researchers have done lots of efforts and made great achievements. Tan et al. [10,11] documented the plane surface flow suddenly set in motion and unsteady flows between two parallel plates of a fractional Maxwell fluid model respectively. Exact solutions of velocity were obtained by using Laplace transform and Fourier transform of fractional calculus. The influence of the fractional order of the constitutive relationship on the flow field was significant. Vieru et al. [12] analyzed the unsteady flow of viscoelastic Maxwell model with the fractional derivative between two side walls over an infinite plate in the absence of

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side walls. And analytical solutions for the velocity field were derived. Further, flow of a fractional Maxwell fluid between two parallel plates was studied by Li et al. [13], in which the governing equations were solved numerically combining non-shifted Grünwald formula with finite difference method. Other researches concerning this field contain unsteady natural convection [14], viscoelastic nanofluid [15], porous medium [16] and Hall-MHD system [17–19] etc.

In recent years, it is necessary to discuss the viscoelastic fluid flow in duct for its extensive application in oil exploitation, blood flow and the processing of polymers in pipes. Qi et al. [20] investigated the unsteady flow of a fractional Oldroyd-B fluid through a circular microchannel by using the Hankel transform and the Laplace transform. The exact solutions for the velocity distribution were formulated. At the same time, the numerical results were also obtained. Their work can provide a theoretical guidance for the design of microfluidic equipments. Dalal et al. [21] presented numerical study of driven flows of shear thinning viscoelastic fluids in rectangular cavities. Athar et al. [22] determined the unsteady flow of a fractional Maxwell fluid driven by an infinite circular cylinder. The exact solutions were obtained by means of the integral transform.

MHD flow has been widely employed in energy generation and geophysical fluid dynamics [23–25]. The exact solution of MHD boundary layer flow of an upper-convected Maxwell fluid was got with the homotopy analysis method by Hayat et al. [26]. Dousset et al. [27] and Chatterjee et al. [28] carried out the MHD vortex dynamics in a square duct subjected to a strong externally imposed axial magnetic field respectively. Rashidi [29] developed a novel analytical method (DTM-Pade) to solve MHD boundary-layer equations. Zhou et al. [30,31] established energy decay, well-posedness and blow-up criteria for the incompressible MHD equation with the Hall effect respectively. Many researchers studied the MHD flow in following aspects such as second Stokes flow [32], stagnation-point flow in porous media [33], a slip moving plate [34] and nanofluid boundary layer flow [35] etc.

After the flow control equations of viscoelastic fluid are established, how to get the solutions is a key. Recently, the algorithm of fractional differential equations has made great progress. Liu et al. [36–50] presented a series of new numerical methods and techniques, such as fractional method of lines [36–38], Euler approximation [39,40], fractional predictor–corrector method [41], semi-implicit alternating direction method [42], finite difference method [43–45], finite volume method [46], meshless method [47], spectral method [48], finite element method [49,50] and so on. Luchko [51] firstly developed the operational calculus of Mikusiński's type for the Caputo fractional differential operator, which was adopted to obtain analytical solution of an initial value problem for multi-term time fractional equation. With the method of modified separating variables, Liu et al. [52–55] derived analytical solutions of the time-fractional telegraph equation, fractional diffusion-wave equation, advection–diffusion equation and multi-term time fractional differential equation respectively. However, the above studies of generalized multi-term time and space fractional partial differential equations are mainly concerned on one-dimensional case. A few researches focus on the two-dimensional flow of the fractional viscoelastic fluid. The 2D flow of the fractional Maxwell fluid in a rectangular pipe caused by the external pressure gradient has rarely been reported because of the complexity of the calculation, although it is often encountered in the actual production.

In this paper, we presented analytical and numerical solutions of unsteady two-dimensional momentum equation of a fractional Maxwell fluid in a rectangular pipe in the presence of external pressure gradient. A new separation of variables method [54] is employed to solve the governing equation. Meanwhile, numerical solution of the time fractional problem has been obtained by an effective implicit difference method. The effects of pertinent parameters on the velocity field are discussed graphically in detail.

2. Governing equations

The upper-converted derivative constitutive model of an incompressible fractional Maxwell fluid is given as follows

$$\mathbf{T} = -P\mathbf{I} + \mathbf{S},\ \mathbf{S} + \lambda^{\alpha}(D_t^{\alpha}\mathbf{S} + \mathbf{V}\cdot\nabla\mathbf{S} - \mathbf{LS} - \mathbf{SL}^{T}) = \mu\mathbf{A}, \tag{2.1}$$

where $\mathbf{T}$, $-P\mathbf{I}$, $\mathbf{S}$ and $\mathbf{V}$ are the Cauchy stress tensor, indeterminate spherical stress, extra-stress tensor and velocity respectively, λ and $\mathbf{L}$ denote the relaxation time and the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ and μ are the first Rivlin–Ericksen tensor and the dynamic viscosity, α is fractional parameter and satisfies $0 < \alpha \le 1$, D_t^{α} is the Caputo time fractional derivative of arbitrary order α, which is defined as [56]

$$D_t^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\eta)^{n-\beta-1}f^{(n)}(\eta)d\eta,\ t \ge 0, \tag{2.2}$$

where $n-1 < \beta \le n$, $\Gamma(\cdot)$ is the Gamma function. It should be noted that while $\alpha = 1$, Eq. (2.2) is simplified as the ordinary Maxwell model.

We will consider the two-dimension flow of the fractional Maxwell fluid in the horizontal rectangular pipe. Actual configuration of the problem is shown in Fig. 1. The fluid motion is unidirectional, with x-axis being the coordinate along axial direction and y- and z-axes perpendicular to the pipe. The form of the velocity field is

$$\mathbf{V} = (u,\ v,\ w) = [u(y,\ z,\ t), 0, 0]. \tag{2.3}$$

The fluid is permeated by a uniform magnetic field B_0 applied in the positive y-direction. The magnetic body force is represented by $\sigma B_0^2 u$ for a low magnetic Reynolds number, where σ is the electrical conductivity of the fluid. And there