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Finite element analysis of nonlocal coupled parabolic problem using Newton's method

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ABSTRACT

In this article, we propose finite element method to approximate the solution of a coupled nonlocal parabolic system. An important issue in the numerical solution of nonlocal problems while using the Newton's method is related to its structure. Indeed, unlike the local case the Jacobian matrix is sparse and banded, the nonlocal term makes the Jacobian matrix dense. As a consequence computations consume more time and space in contrast to local problems. To overcome this difficulty we reformulate the discrete problem and then apply the Newton's method. We discuss the well-posedness of the weak formulation at continuous as well as at discrete levels. We derive *a priori* error estimates for both semi-discrete and fully-discrete formulations. Results based on usual finite element method are provided to confirm the theoretical estimates.

1. Introduction

Partial differential equations have many applications in various field of science and engineering. There are many real life problems which involve more than one unknown functions. In this work we consider the following nonlocal coupled parabolic boundary value problem [1,2] with unknowns u and v.
Find $\{u, v\} = \{u(x,t), v(x,t)\}$, $x \in \Omega$ and $t > 0$ such that

$$u_t - a_1(l_1(u), l_2(v))\Delta u = f_1(x,t) \quad \text{in} \quad \Omega \times (0,T), \tag{1.1a}$$
$$v_t - a_2(l_1(u), l_2(v))\Delta v = f_2(x,t) \quad \text{in} \quad \Omega \times (0,T), \tag{1.1b}$$
$$u(x,t) = v(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.1c}$$
$$u(x,0) = u_0(x) \quad \text{on} \quad \Omega, \tag{1.1d}$$
$$v(x,0) = v_0(x) \quad \text{on} \quad \Omega, \tag{1.1e}$$

where Ω is a convex polygonal or polyhedral domain in $\mathbb{R}^d (d \geq 1)$ with boundary $\partial\Omega$ and $f_1(x,t)$, $f_2(x,t)$, $u_0(x)$, $v_0(x)$ are given functions. Here, a_1, a_2 are Lipschitz continuous positive functions and l_1, l_2 are continuous linear forms. In [1,2], authors have considered a similar reaction–diffusion model with two additional nonlinearities $\lambda_1|u|^{p-2}u$, $\lambda_2|v|^{p-2}v$ and $p > 1, \lambda_1, \lambda_2 \geq 0$. Problem (1.1a)–(1.1e) is a special case of the nonlinear coupled system of reaction–diffusion model considered in [1,2].

During past few years many authors have paid attention towards the study of nonlocal elliptic and parabolic problems. In [3–5], authors studied the existence and uniqueness of solutions for the following elliptic nonlocal problem.

$$\begin{aligned} -M(\|\nabla u\|^2)\Delta u &= f(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^d (d \geq 1)$ is bounded domain with Lipschitz boundary $\partial\Omega$, $\|\cdot\|$ denotes the $L^2(\Omega)$ norm, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}_+$ are given functions. In [3], author made a successful attempt to study finite element method for the problem (1.2). Author also addressed the key issue of sparsity of Jacobian matrix obtained, while applying Newton's method to nonlocal elliptic problems of Kirchhoff type.

In [6–8], authors studied existence and uniqueness of the solution for the following nonlocal problem

$$\begin{aligned} u_t - M(l(u))\Delta u &= f(x,t) \quad \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{1.3}$$

where M is some function from $\mathbb{R} \to \mathbb{R}_+$ and $l : L^2(\Omega) \to \mathbb{R}$ is continuous linear form. In [9,10], authors considered the same problem with $l(u)$ as nonlinear function defined by $l(u) = \int_\Omega |\nabla u|^2 dx$. In [10,11], authors made first successful attempt to study finite element scheme together with Newton's method for the numerical approximation to parabolic nonlocal problem (1.3). They also addressed the key issue of sparsity of Jacobian matrix obtained, while applying Newton's method to scalar nonlocal parabolic problems.

Authors in [12], investigated the existence, uniqueness and continuity (with respect to initial values) of the weak solution for the following reaction–diffusion model

$$\begin{aligned} u_t - a(l(u))\Delta u + |u|^{p-2}u &= f(u) \quad \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega. \end{aligned} \tag{1.4}$$

They also investigated exponential stability of the weak solutions and existence of global attractor ($p \geq 2$).
As a motivation for real life problems involving more than one unknown function, authors in [13], consider an example of island with two types of species: Rabbits and Foxes. In this example one species plays the role of predator while the other plays the role of prey. If we are interested to model the population growth of both species, then we have to keep in mind that if, for example, the population of the foxes increases, then the rabbit population will be affected. So, the rate of change of the population of one type will depend on the actual population of the other type. For example, in the absence of the rabbit population, the fox population will decrease (and fast) to face a certain extinction. Something that most of us would like to avoid. In this case, if the difference among the two populations tends to zero, then there is one natural control for the species. In [13], authors also studied existence and uniqueness of the solution for coupled nonlocal system of the following form:

$$\begin{aligned} u_t - a(l(u))\Delta u + f(u-v) &= \alpha(u-v) \quad \text{in} \quad \Omega \times (0,T), \\ v_t - a(l(u))\Delta v - f(u-v) &= \alpha(v-u) \quad \text{in} \quad \Omega \times (0,T), \end{aligned} \tag{1.5}$$

where $a(\cdot) > 0$, l is continuous linear form, $\alpha > 0$ is a parameter and f is Lipschitz continuous function. Here, in this problem (1.5) u and v could describe the densities of two populations that interact through a parameter α.

In [14], authors investigated the propagation of an epidemic disease modeled by a system of three partial differential equations, where ith equation is of the form

$$(u_i)_t - a_i\left(\int_\Omega u_i dx\right)\Delta u_i = f_i(u_1, u_2, u_3), \tag{1.6}$$

in a domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$. They used finite volume scheme, established the existence of solutions and its convergence to weak solution of the partial differential equation (1.6).

Recently, in [1,2] authors have considered the following coupled nonlocal parabolic system

$$\begin{aligned} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u &= f_1(x,t) \quad \text{in} \quad \Omega \times (0,T), \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v &= f_2(x,t) \quad \text{in} \quad \Omega \times (0,T), \\ u(x,t) = v(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega, \\ v(x,0) &= v_0(x) \quad \text{on} \quad \Omega, \end{aligned} \tag{1.7}$$

where $p > 1, \lambda_1, \lambda_2 \geq 0$. In this reaction–diffusion model, u and v could describe the densities of two populations that interact through the functions a_1 and a_2. In this work authors have investigated the existence and uniqueness of the strong global solution by using Faedo–Galerkin method, Aubin–Lions lemma and have given important results on polynomial and exponential decay of the solution in finite time. For obtaining numerical solution of (1.7), authors in [2] used Euler–Galerkin finite element method and provided optimal rates of convergence in $L^2(\Omega)$ norm only. For linearizing the discrete problem, authors in [2] used fixed point (Picard's) method. Although for simplicity, in our work we have analyzed problem (1.7) for the special case $\lambda_1 = 0 = \lambda_2$, but the proposed methodology can be extended to the general case $\lambda_1, \lambda_2 \geq 0$.