ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the blow-up criterion for the quasi-geostrophic equations in homogeneous Besov spaces

Zujin Zhang

School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, Jiangxi, PR China

ARTICLE INFO

Article history:
Received 8 July 2017
Received in revised form 17 September 2017
Accepted 28 October 2017
Available online xxxx

Keywords:
Quasi-geostrophic equations
Blow-up criteria
Besov spaces

ABSTRACT

In this paper, we consider the blow-up criterion for the quasi-geostrophic equations with dissipation Λ^γ $(0<\gamma<1)$. By establishing a new trilinear estimate, we show that if

$$\theta \in L^{\frac{\gamma}{\gamma+s-1}}(0,T;\dot{B}^s_{\infty,\infty}(\mathbb{R}^2))$$

for some $s\in\left(1-\frac{\gamma}{2},1\right)$, then the solution can be extended smoothly past T. This improves and extends the corresponding results in Dong and Pavlović (2009) ([32]) and Yuan (2010).

1. Introduction

This paper considers the quasi-geostrophic equations

$$\begin{cases}\partial_t\theta+(\boldsymbol{u}\cdot\nabla)\theta+\kappa\Lambda^\gamma\theta=0,\\ \boldsymbol{u}=-\nabla^\perp\Lambda^{-1}\theta=-\boldsymbol{\mathcal{R}}^\perp\theta=(\mathcal{R}_2\theta,-\mathcal{R}_1\theta),\\ \theta(0)=\theta_0,\end{cases} \tag{1}$$

where θ is a scalar real value function representing the potential temperature, $\boldsymbol{u}$ is the fluid velocity field, $\kappa>0$ is the dissipation coefficient, Λ^γ is given through Fourier transform $\mathscr{F}$ as

$$\mathscr{F}(\Lambda^\gamma f)(\xi)=|\xi|^\gamma\mathscr{F}(f)(\xi), \tag{2}$$

$\nabla^\perp=(-\partial_2,\partial_1)=\left(-\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\right)$ and $\boldsymbol{\mathcal{R}}=(\mathcal{R}_1,\mathcal{R}_2)$ is the two dimensional Riesz transforms.

The quasi-geostrophic system is an important model in atmospheric and oceanic fluid (see [1,2]). When $\gamma=1$, system (1) shares many similar features to the three-dimensional Navier–Stokes equations. In fact, applying the operator $\nabla^\perp$ to $(1)_1$, and noticing that $\nabla\cdot\boldsymbol{u}=0$, we deduce

$$\partial_t\nabla^\perp\theta+\kappa\Lambda^\gamma\nabla^\perp\theta+(\boldsymbol{u}\cdot\nabla)\nabla^\perp\theta-(\nabla^\perp\theta\cdot\nabla)\boldsymbol{u}=\boldsymbol{0}. \tag{3}$$

We then see readily that $\nabla^\perp\theta$ plays the role of vorticity in the Navier–Stokes equations with fractional dissipation (see [1,3]). Thus the cases $0<\gamma<1$, $\gamma=1$ and $\gamma>1$ are called the supercritical, critical and subcritical cases respectively. The existence of a global weak solution to (1) was established in [4]. For the strong solutions, global existence was already known in the subcritical case [5], and many efforts have been devoted to the critical case , see [6–9]. However, the global regularity in the supercritical case remains open. One is referred to [10–16] for small data global existence results. Consequently, it is

E-mail address: zhangzujin@gnnu.cn.

https://doi.org/10.1016/j.camwa.2017.10.031
0898-1221/

Please cite this article in press as: Z. Zhang, On the blow-up criterion for the quasi-geostrophic equations in homogeneous Besov spaces, Computers and Mathematics with Applications (2017), https://doi.org/10.1016/j.camwa.2017.10.031.

natural to characterize the first finite blow up time T^* using suitable norms of the solutions. Let us list some progresses in this respect (see [17–25] for further investigations):

(1) Constantin–Majda–Tabak [1] established the following Beale–Kato–Majda criterion (see [26])

$$\int_0^{T^*} \left\|\nabla^{\perp}\theta(\tau)\right\|_{L^\infty} \mathrm{d}\tau = \infty; \tag{4}$$

(2) Chae [3] showed the fundamental Serrin type regularity condition

$$\int_0^{T^*} \left\|\nabla^{\perp}\theta(\tau)\right\|_{L^q}^p \mathrm{d}\tau = \infty, \quad \frac{\gamma}{p} + \frac{2}{q} = \gamma, \quad \forall\, \frac{2}{\gamma} < q < \infty; \tag{5}$$

(3) Dong–Chen [27] then extend (5) to homogeneous Besov spaces $\dot{B}^0_{q,\infty}(\mathbb{R}^2) \supset L^q(\mathbb{R}^2)$ (see [28] for the definition and its fine properties) as

$$\int_0^{T^*} \left\|\nabla^{\perp}\theta(\tau)\right\|_{\dot{B}^0_{q,\infty}}^p \mathrm{d}\tau = \infty, \quad \frac{\gamma}{p} + \frac{2}{q} = \gamma, \quad \forall\, \frac{4}{\gamma} \leqslant q \leqslant \infty, \tag{6}$$

with the remaining cases $\frac{2}{\gamma} < q < \frac{4}{\gamma}$ covered in [29];

(4) Yuan [30] then invoke the Hölder type inequalities in homogeneous Besov spaces to derive the following blow-up criterion

$$\int_0^{T^*} \left\|\nabla^{\perp}\theta(\tau)\right\|_{\dot{B}^{-\delta-\gamma/2}_{\infty,\infty}}^{\frac{2\gamma}{\gamma-2\delta}} \mathrm{d}\tau = \infty \quad \forall\, 0 < \delta < \frac{\gamma}{2}, \tag{7}$$

which is equivalent to saying that

$$\int_0^{T^*} \|\theta(\tau)\|_{\dot{B}^s_{\infty,\infty}}^{\frac{\gamma}{\gamma+s-1}} \mathrm{d}\tau = \infty, \quad \forall\, 1-\gamma < s < 1-\frac{\gamma}{2} \tag{8}$$

in view of the fact that $\|\nabla f\|_{\dot{B}^s_{p,q}}$ and $\|f\|_{\dot{B}^{s+1}_{p,q}}$ are equivalent norms (see [28]);

(5) on the other hand, Dong–Pavlović [31] considered the regularity criterion in inhomogeneous Besov spaces:

$$\int_0^{T^*} \|\theta(\tau)\|_{B^s_{q,\infty}}^p \mathrm{d}\tau = \infty, \quad \frac{\gamma}{p} - s + \frac{2}{p} = \gamma - 1, \quad \forall\, \{p, q\} \subset [2, \infty), \tag{9}$$

which was further generalized in [32, Theorem 3.5] as

$$\int_0^{T^*} \|\theta(\tau)\|_{B^s_{\infty,\infty}}^{\frac{\gamma}{\gamma+s-1}} \mathrm{d}\tau = \infty, \quad \forall\, 1-\gamma < s \leqslant 1. \tag{10}$$

These above mentioned results motivate our study. Our aim is to improve (10) from inhomogeneous Besov spaces to homogeneous ones. Notice that the cases $1-\gamma < s < 1-\frac{\gamma}{2}$ and $s = 1$ was already achieved in (7) and (6) respectively. So we shall consider the case $1-\frac{\gamma}{2} \leqslant s < 1$ only. Our result reads as follows.

Theorem 1. *Let $\theta_0 \in H^3(\mathbb{R}^2)$. Assume that $\theta \in C([0, T^*); H^3(\mathbb{R}^2))$ is the unique local smooth solution of (1) with $T^* < \infty$ being the first blow up time. Then it is necessary that*

$$\int_0^{T^*} \|\theta(\tau)\|_{\dot{B}^s_{\infty,\infty}}^{\frac{\gamma}{\gamma+s-1}} \mathrm{d}\tau = \infty, \quad \forall\, 1-\frac{\gamma}{2} < s < 1. \tag{11}$$

Remark 2. Since $B^s_{p,q}(\mathbb{R}^2) = L^p(\mathbb{R}^2) \cap \dot{B}^s_{p,q}(\mathbb{R}^2)$ for any $s > 0$ and $1 \leqslant p, q \leqslant \infty$ (see [33, Theorem 6.3.2]), we see our result improves (10) indeed. However, due to our utilization of Lemma 3, we could not show the following blow up criterion

$$\int_0^{T^*} \|\theta(\tau)\|_{\dot{B}^{1-\frac{\gamma}{2}}_{\infty,\infty}}^2 \mathrm{d}\tau = \infty \tag{12}$$

at this moment. We hope we can investigate this issue in the future.

During the proof of Theorem 1 in Section 2, we shall use the following key trilinear estimates.

Lemma 3. *Let $\alpha > 0$, $1-\alpha < s < 1$, and $1 \leqslant k \leqslant n \in \mathbb{N}$. Then for any $f \in \dot{B}^s_{\infty,\infty}(\mathbb{R}^n)$, $g, h \in \dot{H}^\alpha(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \partial_k f \cdot gh \,\mathrm{d}x \leqslant C\|f\|_{\dot{B}^s_{\infty,\infty}} \|g, h\|_{L^2}^{2-\frac{1-s}{\alpha}} \|\Lambda^\alpha g, \Lambda^\alpha h\|_{L^2}^{\frac{1-s}{\alpha}}. \tag{13}$$