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Regularity criteria for the three dimensional Ericksen–Leslie system in homogeneous Besov spaces

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In this paper, we establish some regularity criteria involving homogeneous Besov spaces for both the simplified and the general three dimensional Ericksen–Leslie system. This improves many previous results, and can be viewed as the ultimate optimal regularity criterion in the Besov space framework.

1. Introduction

The hydrodynamic theory of liquid crystals was initialled by Ericksen and Leslie [1–4]. Due to its complexity, Lin-Liu [5] introduced the standard penalty approximation to simplify it as

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla P = -\nabla\cdot[\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}], \\ \partial_t \boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} = \triangle \boldsymbol{d} - \boldsymbol{f}(\boldsymbol{d}), \\ \operatorname{div} \boldsymbol{u} = 0, \\ (\boldsymbol{u}, \boldsymbol{d})|_{t=0} = (\boldsymbol{u}_0, \boldsymbol{d}_0), \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)^t$ is the fluid velocity field, $\boldsymbol{d} = (d_1, d_2, d_3)^t$ models the (averaged) macroscopic/continuum molecule orientation, P is the pressure arising from the usual assumption of incompressibility $\operatorname{div} \boldsymbol{u} = 0$, and

$$\boldsymbol{f}(\boldsymbol{d}) = \frac{1}{\eta^2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}, \quad (\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d})_{ij} = \partial_i d_k \partial_j d_k \tag{2}$$

with $\eta > 0$. Here and hereafter, we shall use the summation convention over repeated indices.

The global existence of a weak solution and the local-in-time strong solutions to (1) was already estimated in [5]. However, the global regularity remains unanswered. Fan–Guo [6] first showed the following two fundamental Serrin type regularity criteria involving $\boldsymbol{u}$ or $\nabla \boldsymbol{u}$:

$$\boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leqslant \infty, \tag{3}$$

$$\nabla \boldsymbol{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leqslant \infty, \tag{4}$$

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with the margin cases $\boldsymbol{u}\in L^2(0,T;L^\infty(\mathbb{R}^3))$ and $\nabla\boldsymbol{u}\in L^1(0,T;L^\infty(\mathbb{R}^3))$ extended to be

$$\boldsymbol{u}\in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)),\quad \nabla\boldsymbol{u}\in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)) \tag{5}$$

by Fan–Ozawa [7]. Here, $\dot{B}^s_{p,q}(\mathbb{R}^3)$ with $s\in\mathbb{R}$ and $1\leqslant p,q\leqslant\infty$ is the homogeneous Besov spaces, one is referred to [8] for definitions, fine properties and its utilization in fluid dynamical systems.

The first purpose of this paper is to show an intermediate regularity criterion.

Theorem 1. *Let $\boldsymbol{u}_0\in H^1(\mathbb{R}^3)$ with $\operatorname{div}\boldsymbol{u}_0=0$, $\boldsymbol{d}_0\in H^2(\mathbb{R}^3)$. Assume that $(\boldsymbol{u},\boldsymbol{d})$ is the corresponding local smooth solution pair to (1) on $[0,T)$. If additionally,*

$$\boldsymbol{u}\in L^{\frac{2}{1+r}}(0,T;\dot{B}^r_{\infty,\infty}(\mathbb{R}^3)) \tag{6}$$

for some $0<r<1$, then the solution can be extended smoothly past T.

Remark 2. The margin case $r=0$ or $r=1$ is just (5), which was shown in [7]. This is why we call our result is intermediate.

The main idea in proving Theorem 1 is the following trilinear estimates, which could have its own interest.

Lemma 3. *For $f\in\dot{B}^r_{\infty,\infty}(\mathbb{R}^3)$, $g,h\in H^1(\mathbb{R}^3)$ and any $\varepsilon>0$, $0<r<1$, $k\in\{1,2,3\}$, we have*

$$\int_{\mathbb{R}^3}\partial_k f\cdot gh\,\mathrm{d}x\leqslant C\|f\|^{\frac{2}{1+r}}_{\dot{B}^r_{\infty,\infty}}\|(g,h)\|^2_{L^2}+\varepsilon\|\nabla(g,h)\|^2_{L^2}. \tag{7}$$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}^3}\partial_k f\cdot gh\,\mathrm{d}x&=-\int_{\mathbb{R}^3}f\cdot\partial_k(gh)\,\mathrm{d}x=-\int_{\mathbb{R}^3}\Lambda^r f\cdot\Lambda^{-r}\partial_k(gh)\,\mathrm{d}x\quad\left(\Lambda=(-\triangle)^{\frac12}\right)\\
&\leqslant C\left\|\Lambda^r f\right\|_{\dot{B}^0_{\infty,\infty}}\left\|\Lambda^{-r}\partial_k(gh)\right\|_{\dot{B}^0_{1,1}}\quad\text{(by [8, Proposition 2.29])}\\
&\leqslant C\|f\|_{\dot{B}^r_{\infty,\infty}}\|gh\|_{\dot{B}^{1-r}_{1,1}}\quad\text{(by [8, Lemma 2.1])}\\
&\leqslant C\|f\|_{\dot{B}^r_{\infty,\infty}}\left(\|g\|_{L^2}\|h\|_{\dot{B}^{1-r}_{2,1}}+\|g\|_{\dot{B}^{1-r}_{2,1}}\|h\|_{L^2}\right)\\
&\qquad\text{(by analogues of [8, Corollary 2.54])}\\
&\leqslant C\|f\|_{\dot{B}^r_{\infty,\infty}}\left(\|g\|_{L^2}\|h\|^r_{\dot{B}^0_{2,\infty}}\|h\|^{1-r}_{\dot{B}^1_{2,\infty}}+\|g\|^r_{\dot{B}^0_{2,\infty}}\|g\|^{1-r}_{\dot{B}^1_{2,\infty}}\|h\|_{L^2}\right)\\
&\qquad\text{(by [8, Proposition 2.22])}\\
&\leqslant C\|f\|_{\dot{B}^r_{\infty,\infty}}\left(\|g\|_{L^2}\|h\|^r_{L^2}\|\nabla h\|^{1-r}_{L^2}+\|g\|^r_{L^2}\|\nabla g\|^{1-r}_{L^2}\|h\|_{L^2}\right)\\
&\qquad\text{(by [8, Proposition 2.39])}\\
&\leqslant C\|f\|_{\dot{B}^r_{\infty,\infty}}\|(g,h)\|^{1+r}_{L^2}\|\nabla(g,h)\|^{1-r}_{L^2}\\
&\leqslant C\|f\|^{\frac{2}{1+r}}_{\dot{B}^r_{\infty,\infty}}\|(g,h)\|^2_{L^2}+\varepsilon\|\nabla(g,h)\|^2_{L^2}.\quad\square
\end{aligned}$$

When $\eta\to0^+$, system (1) reduces to

$$\begin{cases}\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\triangle\boldsymbol{u}+\nabla P=-\nabla\cdot(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}),\\ \partial_t\boldsymbol{d}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}=\triangle\boldsymbol{d}+|\nabla\boldsymbol{d}|^2\boldsymbol{d},\\ \operatorname{div}\boldsymbol{u}=0,\quad|\boldsymbol{d}|=1,\\ (\boldsymbol{u},\boldsymbol{d}_0)|_{t=0}=(\boldsymbol{u}_0,\boldsymbol{d}_0).\end{cases} \tag{8}$$

This system recently attracts many authors' attention. Fan–Guo [6] showed the fundamental Serrin type regularity criterion:

$$\begin{aligned}\boldsymbol{u}&\in L^p(0,T;L^q(\mathbb{R}^3)),\quad\frac2p+\frac3q=1,\quad q>3,\\ \nabla\boldsymbol{d}&\in L^r(0,T;L^s(\mathbb{R}^3)),\quad\frac2r+\frac3s=1,\quad s>3.\end{aligned} \tag{9}$$

Later on, Fan–Gao–Guo [9] established the following two blow-up criteria:

$$\boldsymbol{u},\ \nabla\boldsymbol{d}\in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)), \tag{10}$$

$$\boldsymbol{\omega}\overset{\text{def}}{=}\nabla\times\boldsymbol{u},\ \triangle\boldsymbol{d}\in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)). \tag{11}$$