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Error estimates of the space–time spectral method for parabolic control problems☆

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ABSTRACT

This work is devoted to investigate the spectral approximation of optimal control of parabolic problems. The space–time method is used to boost high-order accuracy by applying dual Petrov–Galerkin spectral scheme in time and spectral method in space. The optimality conditions are derived, and the a priori error estimates indicate the convergence of the proposed method. Numerical tests confirm the theoretical results, and show the efficiency of the method.

1. Introduction

In many applications, the optimal control of PDEs is the ultimate goal for us. One can find a series of real-life examples on this aspect, such as the optimal control of flow motion with the purpose of achieving some desired objective, the optimal strategy of a thermal treatment in cancer therapy, and the optimal shape design of an aircraft. Here, we refer to [1–3] for some concrete applications of optimal control problems. There exists a vast literature about theoretical aspects for optimal control problems governed by PDEs. Extensive research has been carried out on the existence and regularity of optimal solution, optimality conditions and Lagrange multipliers in [4–9].

Wide applications and fruitful theoretical results lead to an extensive study for designing efficient numerical schemes for optimal control problems. As we know, adaptive finite element method has been widely applied in optimal design problems in the last two decades. From among the many contributions we mention the systematic introduction in [10]. In this work, the authors provided the finite schemes and optimality conditions, derived a priori error and a posteriori error estimates, investigated adaptive process and superconvergence analysis, and designed fast algorithms for discrete systems of optimal control. We refer to [11–16] for more work on finite element and adaptive finite element approximation for optimal control problems. The mixed finite element method was used for solving control problems in [17,18], and the semi-smooth Newton method was employed for approximating state-constrained control problems in [19,20]. The numerical strategies of primal–dual active set algorithm and augmented Lagrangian method were proposed in [21,22]. Recently the spectral method has gained increasing success for dealing with optimal control. The Legendre–Galerkin spectral method was investigated to solve the control problems with integral control constraint in [23,24], where both a priori error estimates and a posteriori error estimates were derived, and the theoretical results were confirmed by some numerical examples. Moreover, the a

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priori error estimates were provided for state constrained problems governed by the first biharmonic equation in [25]. For time-dependent cases, the a priori and a posteriori error estimates were established in [26] and [27] respectively for optimal control of time fractional diffusion equation by spectral method.

However, there is a lack on space–time spectral method discretization for parabolic control problems. In this paper, we consider the dual Petrov–Galerkin spectral scheme in time and spectral method in space for the following optimal control problem of parabolic equation:

$$\begin{aligned} &\min_{u\in U_{ad}} J(u,y) = \frac{1}{2}\int_0^T \|y-z_d\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(\Omega)}dt,\\ &\partial_t y - \Delta y = f+u \quad x\in\Omega,\ t\in(0,T),\\ &y|_{\partial\Omega} = 0 \quad t\in(0,T],\\ &y(x,0) = s(x) \quad x\in\bar{\Omega}, \end{aligned} \tag{1.1}$$

where U_{ad} is a closed convex set in control space U and $s(x) = 0$ on the boundary of the domain Ω. The details will be specified in the next section.

On numerical approximation of optimal control governed by parabolic equations, many published results can be found, and both finite difference and finite element methods were important ways for time discretization. In respect of finite element discretization in time, the space–time finite element method was used for control of parabolic problems, which was found very efficient in computing optimal control problems of diffusion dominated systems. The a priori error estimates in [28–30] and a posteriori error estimates in [31,32] were reported for parabolic control by discontinuous Galerkin methods in time and usual conforming finite element method in space for state variables. Furthermore, the continuous Galerkin method was used for time discretization in [33], and a convergence order of $\mathcal{O}(k^2)$ in time was allowed by a post-processing where k is the temporal discretization parameter. In respect of finite difference discretization in time, the reliable a posteriori error estimates were derived for optimal control governed by parabolic equations in [34,35]. The authors consider the control of parabolic type in [18], where the finite difference method was employed for time discretization and mixed finite element method for space discretization. As to the spectral method for optimal control of parabolic problems, we refer to [36] due to the rather limited investigation on this aspect. In this work, the authors derived a posteriori error estimates by high-order spectral method in space coupled with a low-order finite difference scheme in time, which leads to a mismatch in accuracy. It is our main goal to change this mismatch and boost a high-order accuracy in time direction in this work.

The outline of this paper is as follows. The spectral scheme and optimality conditions are presented in Section 2. In Section 3, a priori error estimates are investigated. In Section 4, the implementation details of the proposed method for (1.1) and some numerical results are presented to support theoretical analysis.

Let $I=(a,b)$, $\Omega\subset\mathbb{R}^d$ with $d=1,2,3$, and $D=I\times\Omega$. Moreover, let $H^k(\Omega)$ ($H^k_0(\Omega)$) with k being a non-negative integer be usual Sobolev space on Ω with the norm $\|\cdot\|_{k,\Omega}$. Denote by Z a Sobolev space, and define

$$\begin{aligned} H^m(I;Z) &= \left\{u : \int_a^b \|\partial_t^s u(\cdot,t)\|^2_Z dt < \infty, 0\le s\le m\right\},\\ \|u\|_{H^m(I;Z)} &= \left[\sum_{s=0}^m \int_a^b \|\partial_t^s u(\cdot,t)\|^2_Z dt\right]^{\frac{1}{2}}, \quad m\ge 0, \end{aligned}$$

with $L^2(I;Z)=H^0(I;Z)$ and $\partial_t^s u = \frac{\partial^s u}{\partial t^s}$. The operators Δ and ∇ denote the Laplace and the gradient operators respectively with respect to the spatial variable $x\in\Omega$. In addition, C denotes a general positive constant independent of any function and discretization parameter.

2. The spectral approximation and optimality conditions

In this section, we state the space–time spectral discretization of the control problem, and derive the continue and discrete optimality conditions. To simplify the presentation, we let $\Omega=(-1,1)^d$, $d=1,2,3$, and shift the time interval to $I=(-1,1)$ in (1.1). Furthermore, it is reasonable to assume $s(x)\equiv 0$ without loss of generality since it can be handled easily by transformation $\hat{y}=y-s(x)$. Define $U=L^2(\Omega)$, $V=H^1_0(\Omega)$, and

$$\begin{aligned} &X = L^2(I,U), \quad Y = L^2(I,V)\cap H^1(I,U),\\ &{}^E Y = \{y\in Y : y(x,-1)=0,\ \forall x\in\Omega\},\\ &Y^E = \{y\in Y : y(x,1)=0,\ \forall x\in\Omega\}. \end{aligned}$$

Then the control problem becomes: find $(y,u)\in {}^E Y\times X$ such that

$$\begin{aligned} &\min_{u\in U_{ad}}\left\{\frac{1}{2}\int_{-1}^1 \|y-z_d\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_{-1}^1 \|u\|^2_{L^2(\Omega)}dt\right\},\\ &\partial_t y - \Delta y = f+u \quad x\in\Omega,\ t\in I,\\ &y|_{\partial\Omega} = 0 \quad t\in(-1,1],\\ &y(x,-1) = 0 \quad x\in\bar{\Omega}, \end{aligned} \tag{2.1}$$