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Parameterized generalized shift-splitting preconditioners for nonsymmetric saddle point problems☆

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ABSTRACT

To solve nonsymmetric saddle point problems, the parameterized generalized shift-splitting (PGSS) preconditioner is presented and analyzed. The corresponding PGSS iteration method can be applied not only to the nonsingular saddle point problems but also to the singular ones. The convergence and semi-convergence of the PGSS iteration method are discussed carefully. Meanwhile, the spectral properties of the preconditioned matrix and the strategy of the choices of the parameters are given. Numerical experiments further demonstrate that the PGSS iteration method and the PGSS preconditioner are efficient and have better performance than some existing iteration methods and newly proposed preconditioners, respectively, for solving both the nonsingular and singular nonsymmetric saddle point problems.

1. Introduction

Let $A \in \mathbb{R}^{m\times m}$ be a nonsymmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ be a rectangular matrix, and $f \in \mathbb{R}^m$, $g \in \mathbb{R}^n$, with m and n being two given positive integers such that $n \le m$. Here, B^T is the transpose of B. Then we consider the following two-by-two block linear system of the form:

$$\mathcal{A}u = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1}$$

which is a nonsymmetric saddle point problem and widely arises from many applications in scientific and engineering computing. For background information on saddle point problems in scientific and engineering applications, see, e.g., [1–5]. Here and in the sequel, for a symmetric matrix G, $\lambda_{\min}(G)$ represents the minimum eigenvalue of G, and $\rho(G_1)$ stands for the spectral radius of the matrix G_1. $Re(\lambda)$ and $Im(\lambda)$ denote the real part and imaginary part of λ, respectively. We indicate by $(.)^*$ the conjugate transpose of either a vector or a matrix, and denote by $\mathbb{R}^+$ the set of all positive real numbers.

Although the direct methods are very attractive in the form of preconditioners embedded in an iterative framework for the saddle point problems, iteration methods become more efficient than direct methods when the matrices A and B are large and sparse. While the matrix B of (1) is rank deficient, then we call (1) a singular saddle point problem. There are a large number of numerical methods for solving singular saddle point problems in the recent literatures. Zheng et al. [6] and Bai [7] proposed some conditions for the semi-convergence of the generalized successive overrelaxation (GSOR) method

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and the Hermitian and skew-Hermitian splitting (HSS) method, respectively. Very recently, Li and Zhang [8] introduced the triple-parameter modified symmetric SOR (TMSSOR) method for singular saddle point problems, and Huang et al. [9] constructed the generalized Uzawa-single-step HSS (GU-SHSS) method and analyzed the semi-convergence of the proposed method for singular saddle point problems. Based on the positive definite and skew-Hermitian splitting (PSS) iteration scheme and the Uzawa method, Cao and Yi [10] designed the Uzawa-PSS iteration method for solving both the nonsingular and singular nonsymmetric saddle point problems. A number of other efficient iteration methods have been proposed for singular saddle point problems, including the HSS-type methods [11,12], Uzawa-type methods [13,14], the matrix splitting iteration methods [15–17] and so forth.

When the coefficient matrix $\mathcal{A}$ of (1) is nonsingular, i.e., B is of full column rank, many authors have proposed a variety of iteration methods for nonsingular saddle point problems, such as the SOR-like methods [18–21], Uzawa-type methods [18,19,22–24], matrix splitting iteration methods [25–28] and so on. Furthermore, Bai et al. put forward the well-known HSS methods [29] and its variants [30–33]. Very recently, combining the HSS and the shift-splitting [25,34] of a matrix, Zhou et al. [35] proposed the modified shift-splitting (MSS) iteration method as well as the MSS preconditioner for nonsymmetric saddle point problem (1). Zheng and Ma [36] derived the upper and lower triangular (ULT) splitting iteration method for solving the nonsingular saddle point problems lately. In [37], Bai and Benzi newly developed the regularized HSS (RHSS) iteration method for nonsingular saddle point problems by introducing a Hermitian positive semidefinite matrix in the HSS of the matrix $\mathcal{A}$.

Recently, inspired by the shift-splitting preconditioner presented by Bai et al. [34] to solve the non-Hermitian positive definite linear system, Cao et al. [25] presented a shift-splitting (SS) preconditioner of the form

$$\mathcal{P}_{SS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B \\ -B^T & \alpha I \end{pmatrix}$$

for the saddle point problem (1), where α is a positive constant and I is the identity matrix with appropriate dimension. Subsequently, Chen and Ma [26] and Cao et al. [27] replaced the parameter α in (2,2)-block of the SS preconditioner by another parameter β and established the generalized SS (GSS) preconditioner. It is noteworthy that $\mathcal{P}_{SS}$ is a special case of $\mathcal{P}_{GSS}$ when $\alpha = \beta$. Numerical results in [26,27] confirmed that the GSS preconditioner outperforms the SS one. In papers [25–27], it is straightforward to show that the SS and the GSS iteration methods converge unconditionally to the unique solution of the nonsingular saddle point problem (1). Very recently, Shen and Shi in [38] extended the GSS preconditioner for both the nonsingular and singular generalized saddle point problems. To further show the efficiency of the SS preconditioner, the spectral analyses of the shift-splitting preconditioned nonsingular and singular saddle point matrices have been investigated in [39] and [40], respectively.

In order to improve the efficiency of the GSS iteration method for the nonsingular saddle point problems with symmetric positive definite (1,1) parts, Huang and Su [28] newly constructed the modified shift-splitting (MSSP) of the saddle point matrix $\mathcal{A}$:

$$\mathcal{A} = \mathcal{P}_{MSSP} - \mathcal{Q}_{MSSP} = \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \alpha I \end{pmatrix} - \begin{pmatrix} \alpha I + A & B \\ -B^T & \alpha I \end{pmatrix}$$

with $\alpha > 0$ being a constant and I being the identity matrix with appropriate dimension, which induces the modified shift-splitting (MSSP) preconditioner $\mathcal{P}_{MSSP}$. The authors in [28] theoretically verified the unconditional convergence of the corresponding MSSP iteration method and estimated the bounds of the eigenvalues of the iteration matrix of the MSSP iteration method. Numerical experiments illustrated that the MSSP preconditioner outperforms the SS and the GSS preconditioners for the nonsingular saddle point problems with symmetric positive definite (1,1) parts.

To further accelerate the convergence rates of the GSS and the MSSP preconditioned generalized minimal residual (GMRES) methods for the saddle point problems with nonsymmetric positive definite (1,1) parts, we develop and study the parameterized generalized shift-splitting (PGSS) preconditioner by introducing a positive constant t for nonsymmetric saddle point problems in this paper, which not only covers the existing preconditioners in [25–28], but can also generate more new ones. Theoretical analysis also shows that the corresponding splitting iteration method is convergent and semi-convergent under suitable conditions. Besides, we investigate the spectral properties of the corresponding preconditioned matrix and the choices of parameters for the proposed iteration method. Numerical experiments further verify the effectiveness of the PGSS iteration method and the GMRES method with the PGSS preconditioner for solving the nonsymmetric saddle point problems.

The outline of this paper is organized as follows. Section 2 introduces the PGSS iteration method which induces the PGSS preconditioner, and Section 3 derives the convergence properties of the PGSS iteration method. In Section 4, we study the semi-convergence of the PGSS iteration method. The eigenvalue and the eigenvector distributions of the PGSS preconditioned matrix are analyzed in Section 5. The discussion of the choices of the parameters of the PGSS iteration method and the PGSS preconditioner are exhibited in Section 6. In Section 7, numerical experiments are presented to illustrate the effectiveness of the PGSS iteration method and the PGSS preconditioned GMRES method for solving both the nonsingular and singular nonsymmetric saddle point problems. Comparisons between the results obtained with the PGSS preconditioner and those obtained with the accelerated HSS (AHSS), semi implicit method for pressure linked equations (SIMPLE)-like (SL) [41] which is a general case of the relaxed deteriorated PSS (RDPSS) preconditioner proposed by Cao et al. in [42], generalized variant of the deteriorated PSS (GVDPSS) [43], GSS, generalized MSS (GMSS) [17] and the MSSP preconditioners are given to show that the PGSS preconditioner is superior to the aforementioned preconditioners. Finally, in Section 8, we end this paper with a few concluding remarks.

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