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Analysis of multiscale mortar mixed approximation of nonlinear elliptic equations

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ABSTRACT

A multiscale mortar mixed finite element method is established to approximate nonlinear second order elliptic equations. The method is based on non-overlapping domain decomposition and mortar finite element methods. The existence and uniqueness of the approximation are demonstrated, and a priori L^2-error estimates for the velocity and pressure are derived. An error bound for mortar pressure is proved. Convergence estimates of the mortar pressure are based on a linear interface formulation having the discrete-pressure dependent coefficient. Optimal order convergence is achieved on the fine scale by a proper choice of mortar space and polynomial degree of approximation. The quadratic convergence of the Newton–Raphson method is proved for the nonlinear algebraic system arising from the mortar mixed formulation of the problem. Numerical experiments are performed to support theoretic results.

1. Introduction

A vast range of phenomena in science and engineering leads to nonlinear partial differential equations, for example, gas dynamics, fluid mechanics, elasticity, etc. Many real world physical systems of interest involve nonlinearity along with heterogeneity, e.g., in fluid flow in porous media the heterogeneity is caused by the formation of medium and it is represented by permeability.

We consider the non-linear second order elliptic problem that models Darcy flow in porous media,

$$\boldsymbol{u} = -K(p)\nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = f \quad \text{in } \Omega, \tag{1.2}$$

$$p = g \quad \text{on } \partial\Omega, \tag{1.3}$$

for domain $\Omega \subset \boldsymbol{R}^d$, $d = 2$ or 3. Here p is the pressure, $\boldsymbol{u}$ is the Darcy velocity, and $K(p)$ represents permeability divided by viscosity. Indeed, this model is a simplified version of steady, single-phase slightly compressible Darcy flows through a porous medium by neglecting gravity effects. We assume that $K(p)$ is twice continuously differentiable with bounded derivatives through second order. Moreover,

$$0 < c_0 \le K(x, p) \le c_1 < \infty.$$

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The variable x will normally be dropped in the notation. In this paper, Dirichlet boundary conditions are considered for simplicity. However, more general conditions can be handled. We assume that the problem is at least $H^{2+\varepsilon}$-regular for some $0 < \varepsilon < 1$, where H^m is the standard Sobolev space. We have $H^{2+\varepsilon}$-regularity, for example, if $f \in H^{\varepsilon}(\Omega)$, $g \in H^{3/2+\varepsilon}(\partial\Omega)$, and Ω is smooth enough (see, [1–3]).

Mixed finite element methods for second order elliptic equations are well known for their local mass conservation property and good approximation of fluxes, whose normal components are continuous across inter-element boundaries. Mixed methods have been successfully used in various fields, especially, fluid flows in porous media (see, [4–8]). Mixed finite element methods have been extensively analyzed and implemented. Many papers and books deal with mixed methods for linear elliptic problems (see, [9–17]). Nonlinear elliptic problems are treated in [18–21].

When domain Ω is large and coefficient $K(p)$ is heterogeneous fluctuating on fine scale, it is difficult to solve (1.1)–(1.3) in a traditional way. The direct discretization needs full fine grid resolution of variation in $K(p)$ over whole of Ω, which returns a huge system of equations. The computational solution of this huge system may be complicated in many cases.

The multiscale finite element methods are turned up as an impressive technique to solve partial differential equations using modern parallel computer system. The main aim of the multiscale methods is to reduce the computational burden. The multiscale finite element methods [22,23] and variational multiscale methods [24–26] were established for a single second order partial differential equation. The variational multiscale [27–29] and multiscale method [30] were developed to treat mixed system of two first order equations.

In both methods, the problem is split into a set of small problems. An appropriate boundary condition is defined for these coarse problems which are solved on fine scale to specify the multiscale finite element basis. These local bases are then used to approximate the global solution. The computational efficiency comes from decomposition of the problem into small subproblems. In [5], Arbogast et al. introduced multiscale mortar mixed method for the linear elliptic problem modeling Darcy flow. This method is based on domain decomposition theory [31] and mortar finite element [6,32]. In the past decade, this approach has been successfully used for the approximation of linear elliptic problem [4–7,33–38]. This approach has been further studied to approximate unsteady single-phase slightly compressible Darcy flows in porous media in [8,39]. In [8], the optimal fine scale convergence for semi-discrete formulation was proved using elliptic projection, and parallel numerical simulations on some multiscale benchmark problems are given to show the efficiency and effectiveness of the method. Recently, in [39], global Jacobian methods are presented to efficiently resolve nonlinear algebraic equations resulting from fully implicit discretization. The implementation of parallel solvers and preconditioners were discussed.

In this paper, we present a generalization of multiscale mortar mixed method of [5] to nonlinear problems. To the best of the authors' knowledge, there is no paper available in the literature addressing unique solvability and convergence of the multiscale mortar mixed method for the steady case, namely, nonlinear elliptic problems under consideration. Therefore, the aim of this paper is to address theoretical issues. First, we demonstrate unique solvability of the problem. Second, we derive optimal convergence error bounds for the mortar variable as well as the pressure and velocity. Third, we prove the quadratic convergence of Newton's method to solve nonlinear algebraic system arising from the mortar formulation.

Regarding the existence of the solution, we exploit the Brouwer fixed point argument following some of the ideas presented in [19,20,40]. The fixed point approach requires construction of a mapping which maps a ball in appropriate function spaces into itself. If h resolves the fine scale and H is the coarse mortar scale, then the radius of the ball in the fixed point mapping depends on two scales h and H. The choice of norms used to measure the distance from the approximation to a projection of the pressure and velocity variable plays a crucial role in proving solvability of the multiscale mortar mixed formulation. Next, regarding convergence, we show that the velocity and pressure error estimates are of order $\mathcal{O}(H^{m+1/2} + h^{k+1})$. Here, m is the degree of polynomial to approximate the coarse grid pressures, and $k,\ l\,(k = l)$ are the order of approximation for velocities and pressures, respectively, on fine grid scale. Note that mortar interface formulations play a crucial role for implementation as well as theory. However, no previous results are available concerning the convergence of the mortar variable in the nonlinear case. This is probably due to the fact that nonlinearities usually do not allow superposition principle commonly used for linear problems. For the sake of analysis, we propose an interface formulation having the discrete-pressure dependent coefficient, which can be viewed as a linear interface problem. With this mortar interface formulation, we are able to prove convergence of the mortar variables via superposition.

The remainder of the paper is organized as follows. We introduce function spaces and some projection operators in the next section. In Section 3, we prove uniqueness and solvability of the problem. A priori error estimates for velocity, pressure and interface mortar variables are derived in Section 4. In Section 5, we formulate Newton's method to solve nonlinear algebraic system arising from the mortar mixed formulation of the model problem. We also prove the quadratic convergence of Newton's method when the Raviart–Thomas space of index $k > 0$. In Section 6, we present results of numerical experiments confirming theoretical results. The paper is concluded in the last section.

2. Preliminary

For any integer $\ell \geq 0$ and $1 \leq q < \infty$, $W^{\ell,q}(\Omega)$ denotes the usual Sobolev space of which norms and seminorms are denoted by $\|\cdot\|_{\ell,q;\Omega}$ and $|\cdot|_{\ell,q;\Omega}$, respectively. Hereafter we will use the notation $\|\cdot\|_{\ell,\Omega}$ if $q = 2$. For $0 \leq s' < \infty$, let $W^{s',q}(\Omega)$, $W^{s',q}(\partial\Omega)$, $H^{s'}(\Omega)$ and $H^{s'}(\partial\Omega)$ denote fractional order Sobolev spaces with norms $\|\cdot\|_{s',q;\Omega}$, $\|\cdot\|_{s',q;\partial\Omega}$, $\|\cdot\|_{s';\Omega}$ and $\|\cdot\|_{s';\partial\Omega}$, respectively. We will omit the subscript Ω unless it is necessary to avoid ambiguity. Finally, $\boldsymbol{H}(div; D)$ is the space of functions with square integrable weak divergence, i.e., $\boldsymbol{H}(div; D) = \{\boldsymbol{v} \in L^2(D)^2 \mid \nabla\cdot\boldsymbol{v} \in L^2(D)\}$.

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