ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Positive steady states in an epidemic model with nonlinear incidence rate

Xiaoyan Gao [a,b], Yongli Cai [b], Feng Rao [c], Shengmao Fu [a,*], Weiming Wang [b,*]

[a] School of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, PR China
[b] School of Mathematical Science, Huaiyin Normal University, Huaian, 223300, PR China
[c] School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing, Jiangsu 211816, PR China

ARTICLE INFO

Article history:
Received 5 March 2017
Received in revised form 16 August 2017
Accepted 17 September 2017
Available online xxxx

Keywords:
Nonconstant positive steady state
Bifurcation
Turing instability
Nonlinear incidence rate

ABSTRACT

In this paper, we investigate a diffusive epidemic model with nonlinear incidence rate under homogeneous Neumann boundary condition. The value of this study lies in two aspects. Mathematically, we show the stability of the constant positive steady state solution, the existence and nonexistence, the local and global structure of nonconstant positive steady states. And epidemiologically, we find that the system exhibits Turing patterns controlled by diffusion.

1. Introduction

Mathematical models have been important tools in analyzing the spread and control of infectious diseases, qualitatively and quantitatively. The research results are helpful for predicting the developing tendencies of the infectious diseases, for determining the key factors of the disease spreading and for seeking the optimum strategies for preventing and controlling the spread of infectious diseases [1–3]. In mathematical modeling for influenza epidemics, susceptible–infected–recovered–(re)susceptible (SIRS) compartmental model is usually a reasonable qualitative description of the evolutionary dynamics of influenza A [4], SARS [5] and others.

In an effort to understand the protection measures and control polices in disease dynamics, Wang [5] proposed a general epidemic model:

$$\begin{cases} \dfrac{dS}{dt} = dN - dS - \beta(I)S + \nu R, \\ \dfrac{dI}{dt} = \beta(I)S - (d + r)I, \\ \dfrac{dR}{dt} = rI - (d + \nu)R, \end{cases} \tag{1}$$

where S, I and R denote the number of susceptible, infections and recovered individuals, respectively, and the population size by N, i.e., $N = S + I + R$. The parameters d, r, ν are all non-negative constants, d is the per capita death rate, r the recovery rate of infectious individuals, ν the rate of removed individuals who lose immunity and return to susceptible class.

* Corresponding authors.
E-mail addresses: 1075943236@qq.com (X. Gao), caiyongli06@163.com (Y. Cai), raofeng2002@163.com (F. Rao), fusm@nwnu.edu.cn (S. Fu), weimingwang2003@163.com (W. Wang).

https://doi.org/10.1016/j.camwa.2017.09.029
0898-1221/© 2017 Elsevier Ltd. All rights reserved.

And $\beta(I)S$ is the transmission of the infection or the incidence rate, which plays a key role in ensuring that the model does give a reasonable description of the disease dynamics.

It is traditionally supposed that the incidence rate of disease (including SARS, influenza A) transmission is bilinear βSI or standard $\frac{\beta SI}{S+I}$ with respect to the number of susceptible individuals $S(t)$ and the number of infective individuals $I(t)$, here β is the transmission rate [1]. As a matter of fact, it is generally difficult to get the details of transmission of infectious diseases, which may vary under different conditions. In addition, choosing generalized incidence rate function may allow the data themselves to flexibly decide the function form of incidence rates in practice [4]. Motivated by [5], $\beta(I)$ can be factorized as $Ig(I)$, where $g(I)$ satisfies following assumptions:

(H1) $g(0) > 0, g(I) > 0$ is bounded for $I > 0$; $Ig(I) \to k$ as $I \to \infty$;

(H2) $(Ig(I))' > 0$ for $I > 0$; there is an $\eta > 0$ such that $g'(I) > 0,\ I \in (0, \eta)$ and $g'(I) < 0,\ I \in (\eta, \infty)$, and such that $g''(I) \le 0$ for $I \in (0, \eta)$.

Since the population size N is a constant $C > 0$, then model (1) becomes

$$\begin{cases} \dfrac{dI}{dt} = Ig(I)(C - I - R) - (d + r)I, \\ \dfrac{dR}{dt} = rI - (d + \nu)R. \end{cases} \tag{2}$$

On the other hand, it is well known that the spatial component of ecological interactions has been identified as an important factor in how ecological communities are functioning and shaped, yet understanding the role of space is challenging both theoretically and empirically [6–8]. More recently, many studies, e.g., Refs. [9–13] and the references therein, show that the spatial epidemic model is an appropriate tool for investigating the fundamental mechanism of complex spatiotemporal epidemic dynamics.

Based on the discussions above, suppose that the infectious I and the recovered individuals R move randomly—described as Brownian random motion [14], and then we propose a simple spatial model corresponding to (2) as follows:

$$\begin{cases} \dfrac{\partial I}{\partial t} - d_1 \Delta I = Ig(I)(C - I - R) - (d + r)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial R}{\partial t} - d_2 \Delta R = rI - (d + \nu)R, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial \mathbf{n}} = \dfrac{\partial R}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ I(x, 0) = I_0(x) > 0, \quad R(x, 0) = R_0(x) \ge 0, & x \in \Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n$, $\mathbf{n}$ is the outward unit normal vector of the boundary $\partial\Omega$ and the homogeneous Neumann boundary condition is being considered. The diffusion coefficients d_1 and d_2 are positive constants, and the initial data $I_0(x)$, $R_0(x)$ are continuous functions. Δ is the Laplacian operator in $\mathbb{R}^n$ for $n \ge 1$, which describes the Brownian random motion.

The main focus of this paper is to study the dynamics of positive steady states of (3), and the steady states problem corresponding to the following elliptic system:

$$\begin{cases} -d_1 \Delta I = Ig(I)(C - I - R) - (d + r)I, & x \in \Omega, \\ -d_2 \Delta R = rI - (d + \nu)R, & x \in \Omega, \\ \dfrac{\partial I}{\partial \mathbf{n}} = \dfrac{\partial R}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

The rest of the paper is organized as follows. In Section 2, we study long time behavior of solutions of model (3), involving the stability of the constant steady state of (2) and (4). In Section 3, we analyze the fundamental estimates of nonconstant positive solutions and by using the energy method, we obtain various results of the nonexistence of nonconstant positive solutions to (4). And in Section 4, the global structure of the steady state bifurcations from simple eigenvalues is established by bifurcation theory, and the local structure of the steady state bifurcations from double eigenvalues is also obtained by the techniques of space decomposition and implicit function theorem.

2. Long time behavior of the solutions

In this section, we will focus on the persistence property and stability of model (3).

2.1. Persistence property

First, we will show that any nonnegative solution $I(x, t)$, $R(x, t)$ of (3) lies in a certain bounded region as $t \to \infty$ for all $x \in \Omega$. Motivated by [15, Proposition 3.1.], we can easily obtain the following lemma.

Please cite this article in press as: X. Gao, et al., Positive steady states in an epidemic model with nonlinear incidence rate, Computers and Mathematics with Applications (2017), https://doi.org/10.1016/j.camwa.2017.09.029.