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An adaptive wavelet collocation method for solving optimal control of elliptic variational inequalities of the obstacle type

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ABSTRACT

This paper presents a fast computational technique based on the wavelet collocation method for the numerical solution of an optimal control problem governed by elliptic variational inequalities of obstacle type. In this problem, the solution divides the domain into contact and noncontact sets. The boundary between the contact and noncontact sets is a free boundary, which is not known a priori and the solution is not smooth on it. Accordingly, a very fine grid is needed in order to obtain a solution with a reasonable accuracy. In this paper, our aim is to propose an adaptive scheme in order to generate an appropriate and economic irregular dyadic mesh for finding the optimal control and state functions. The irregular mesh will be generated such that its density around the free boundary is higher than in other places and high-resolution computations are focused on these zones. To this aim, we use an adaptive wavelet collocation method and take advantage of the fast wavelet transform of compact-supported interpolating wavelets to develop a multi-level algorithm, which generates an adaptive computational grid. Using this adaptive grid takes less CPU time than using a full regular mesh. At each step of the algorithm, the active set method is used for solving the optimality system of the obstacle problem on the adapted mesh. Finally, the numerical examples are presented to show the validity and efficiency of the technique.

1. Introduction

Variational Inequalities (VIs) in function spaces are an important class of nonlinear problems, which find extensive applications in various fields of science. In recent decades, many problems in physics, fluid mechanics, potential theory, and economy have been reformulated in variational inequalities form [1–4]. In addition, in the mathematical literature, variational inequality problems have a close association with various fields like bi-level programming, Optimal Control Problem (OCP) governed by PDE, and free boundary problem [5,6]. Because of these wide applications, studying numerical methods for variational inequalities and also for OCP constrained with VIs has received a great deal of attention.

In this paper, we deal with OCP governed by an Elliptic Variational Inequality (EVI) of the obstacle type. For the existence and uniqueness of the solution, we refer the reader to the classical work in this field by Barbu [7]. One can find a complete discussion of the basic theory of optimal control of VI in [7]. Furthermore, Ito and Kunisch and their coworkers found the first-order optimality conditions in the sequence of their works (see, for instance [5,8,9]). Penalty approach [10,11], Lagrangian method, and augmented Lagrangian method [12,5] are common strategies for deriving optimality conditions of these problems. For example, in [5], the optimality system for OCP governed by EVI is derived by Lagrangian technique

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and the primal–dual active set method is proposed to numerically solve the optimality system. Bergounioux used the augmented Lagrangian method for optimal control of the elliptic obstacle type problem, where he has considered a relaxed complementarity constraint instead of the exact EVI in his approach [12]. Moreover, he and Zidani [13] could derive the well-known Pontryagin maximum principle for optimal control of VI. Ghanem in [14] derived necessary optimality conditions in a different manner and in [15], he and his coworkers provided a numerical method for the derived necessary optimality conditions. Their proposed algorithm is based on the Gauss–Seidel method. Moreover, a description of some other different numerical methods for the solution of VIs and optimal control of VIs can be found in [2] and references therein.

Nevertheless, providing an efficient discretization scheme, in which a relatively small number of nodes are adequate for a high-accuracy solution, is one of the essential strategies employed to solve all such practical problems. In recent decades, adaptive schemes like adaptive finite element method, adaptive spectral method, and adaptive mesh generation have been developed to accomplish this issue. Almost all of these schemes are based on the computation of a posteriori error or the estimation of the solution with empirical criteria [16–18]. Recently, some of them have been implemented for VIs [19–23] and for OCPs [24–27].

However, to the best of our knowledge, adaptive schemes are rarely implemented for optimal control of VIs. A remarkable effort has been made by Hintermüller and coworkers, who have investigated Finite Element Methods (FEM) for OCP constrained with VIs. They applied a goal-oriented adaptive FEM for this class of optimal control in [28]. Moreover, in [29] they implemented standard residual-type as a posteriori error estimator for adaptive mesh refinement of FEM.

Wavelet-based methods have been widely used for the solution of various kinds of engineering problems (see, e.g., [30–32]). Wavelet methods are implemented for VI in some papers, too. For example, the wavelet Galerkin method is proposed for the numerical solution of a class of VI problems in [33]. Rometsch and Urban [34] applied biorthogonal spline wavelets to implement an adaptive wavelet method for solving EVI. In addition, Hegland and Roberts in [35] present a multiscale approximation of EVI via wavelet Galerkin method, based on solving a minimization problem equivalent to EVI.

Among wavelet-based methods, wavelet collocation methods [36] solve the problem in physical space and in contract wavelet Galerkin methods [37,38] solve the problem in the frequency space. Accordingly, wavelet collocation methods are suitable to generate an adaptive mesh for the solution of a problem.

In this work, we will develop an approach based on the Adaptive Wavelet Collocation Method (AWCM) for the numerical solution of Optimal Control of the Obstacle Problem (OC-OP) and we will propose a novel and efficient adaptive method for this type of problems in one and two-dimensional spaces. We use AWCM for the generation of an adaptive grid in order to apply the active set strategy for solving the discretized first order optimality condition obtained for OC-OP.

The remainder of the paper is organized as follows. The next section is devoted to the formulation of the OC-OP and derivation of the first order optimality conditions. Some useful properties of interpolating wavelets and a background of the adaptive wavelet collocation method are mentioned in Section 3. In Section 4, we apply AWCM for OC-OP and present a computational method to solve it. Numerical examples are given in Section 5 to demonstrate the efficiency of the proposed method. Finally, we conclude the article in Section 6.

2. Optimal control of the obstacle problem

2.1. Problem statement

Let $\mathcal{V}$ be a real Hilbert space and $\mathcal{V}^*$ be the dual space of $\mathcal{V}$. Assume that $a(\cdot,\cdot)$ is a bilinear form defined on $\mathcal{V}\times\mathcal{V}$, which is bounded, continuous and coercive. Moreover, let $\mathcal{A}:\mathcal{V}\to\mathcal{V}^*$ be the linear operator associated with the bilinear form, i.e. $a(\varphi,\psi)=(\mathcal{A}\varphi,\psi)_{\mathcal{V}}$, where $(\cdot,\cdot)_{\mathcal{V}}$ denotes the inner product on $\mathcal{V}$. Consider the following optimal control problem constrained with EVI:

$$\text{Minimize}\quad J[y,u] \tag{1a}$$

$$\text{over}\quad (y,u)\in\mathcal{V}\times\mathcal{U}_{ad}$$

$$\text{subject to}\quad a(y,y-v)\le(f+u,y-v)_{\mathcal{V}},\quad \forall v\in K\subset\mathcal{V} \tag{1b}$$

where $\mathcal{U}_{ad}$ is the admissible set of controls and f belongs to $\mathcal{U}_{ad}$. Here, J refers to the objective functional, y and u stand for the state and the control functions, and K is the constrained set, which makes the constraint (1b) to be an elliptic variational inequality. We note that if $K=\mathcal{V}$, then, Eq. (1b) can be converted to $\mathcal{A}y=f+u$ and it cannot be a variational inequality. Accordingly, $\mathcal{K}$ is considered as a pure subset of $\mathcal{V}$.

As a practical case of OCP constrained with EVI, we consider the Optimal Control of Obstacle Problem (OC-OP). The source and mathematical modeling of OC-OP are described briefly in the following.

Let a membrane is stretched over a planar domain Ω and is deflected by some force having pointwise density f. At the boundary $\partial\Omega$, the membrane is fixed and in the interior of Ω the deflection is assumed to be bounded from above by a given obstacle function g. The problem of finding the equilibrium position of the membrane subject to vertical force f is called the *obstacle problem*. In addition, in the OC-OP, the aim is finding control u such that the shape of the membrane subject to force $f+u$ becomes close to a given desired shape function d, while the norm of u is not allowed to become too large. For more details, please see [5,39].