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A two-grid discretization scheme of non-conforming finite elements for transmission eigenvalues ☆

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A B S T R A C T

In this paper, for the Helmholtz transmission eigenvalue problem, we propose a two-grid discretization scheme of non-conforming finite elements. With this scheme, the solution of the transmission eigenvalue problem on a fine grid π_h is reduced to the solution of the primal and dual eigenvalue problem on a much coarser grid π_H and the solutions of two linear algebraic systems with the same positive definite Hermitian and block diagonal coefficient matrix on the fine grid π_h. We prove the resulting solution still maintains an asymptotically optimal accuracy, and we report some numerical examples in two dimension and three dimension on the modified-Zienkiewicz element to validate the efficiency of our approach for solving transmission eigenvalues.

1. Introduction

The transmission eigenvalue problems have important applications in physics and mathematics (see [1–4]). The computation of transmission eigenvalues has attracted the attention of many researchers in recent years. [5] first studied the numerical methods, including Argyris, continuous and mixed finite element methods, for the Helmholtz transmission eigenvalues. Later, [6–18] etc. developed the work in [5] further, among them [9,11,15] studied a two-grid method and multigrid discretizations based on conforming finite element methods. The two-grid discretization scheme is an efficient approach which was first introduced by Xu [19] for nonsymmetric or indefinite problems, and then has been successfully applied to eigenvalue problems (see, e.g., [9,11,15,20–24] and the references therein). In this paper, based on the weak formulation (2.5) proposed in [15,16] we study a new two-grid scheme of non-conforming finite elements to solve the Helmholtz transmission eigenvalue problem. Our work has the following features.

(1) With our two-grid discretization scheme, the solution of the transmission eigenvalue problem on a fine grid π_h is reduced to the solution of the primal and dual eigenvalue problem on a much coarser grid π_H and the solutions of two linear algebraic systems with the same positive definite Hermitian and block diagonal coefficient matrix on the fine grid π_h, and the resulting solution still maintains an asymptotically optimal accuracy. The key to our theoretical analysis is to prove the error estimate in norm $\|\cdot\|_{\mathbf{H}_1}$ is of higher order than that in norm $\|\cdot\|_h$. The difficulty of the proof lies in the nonsymmetry of the right-hand side term of (2.5) which involves derivatives (see Remark 3.1 in [16]). In Section 3 we introduce an auxiliary boundary value problem and overcome this difficulty by the Nitsche technique in a subtle way. Our Theorem 3.1 and Theorem 3.2 are new results for giving error estimates in a continuous norm while the previous results (see [16]) are in a discrete norm for the non-conforming element methods for the transmission eigenvalue problem.

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(2) For fourth order equations in $\mathbb{R}^3$, many non-conforming elements have advantages over conforming finite elements because of relatively less freedom degrees. For instance, Zenicek constructed a conforming tetrahedral finite element with 9 degree of polynomials and 220 nodal parameters [25], while the modified-Zienkiewicz tetrahedral element [26] and the Morley–Zienkiewicz tetrahedral element [25] have only 16 and 20 nodal parameters, respectively. Hence, our two-grid discretization of non-conforming finite element is easy to realize in three dimensions. In Section 5, we report the examples in both two dimension and three dimension. The numerical results indicate that our method is efficient for computing real and complex transmission eigenvalues as expected.

As for the basic theory of finite elements and spectral approximation, we refer to [25,27–31].

Throughout this paper, C denotes a positive constant independent of mesh size h, which may not be the same constant in different places. For simplicity, we use the symbol $a \lesssim b$ to mean that $a \le Cb$.

2. Preliminaries

Consider the Helmholtz transmission eigenvalue problem: Find $k \in \mathbb{C}$, $w, \sigma \in L^2(\Omega)$, $w - \sigma \in H^2(\Omega)$ such that

$$\Delta w + k^2 n w = 0, \quad in\ \Omega, \tag{2.1}$$

$$\Delta \sigma + k^2 \sigma = 0, \quad in\ \Omega, \tag{2.2}$$

$$w - \sigma = 0, \quad on\ \partial\Omega, \tag{2.3}$$

$$\frac{\partial w}{\partial \gamma} - \frac{\partial \sigma}{\partial \gamma} = 0, \quad on\ \partial\Omega, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded simply connected inhomogeneous medium, γ is the unit outward normal to $\partial\Omega$ and the index of refraction $n(x)$ is positive.

Let $W^{s,p}(\Omega)$ denote the usual Sobolev space with norm $\|\cdot\|_{s,p}$, $H^s(\Omega) = W^{s,2}(\Omega)$, and $\|\cdot\|_{s,2} = \|\cdot\|_s$, $H^0(\Omega) = L^2(\Omega)$ with the inner product $(u, v)_0 = \int_\Omega u\overline{v}dx$. Denote $H_0^2(\Omega) = \{v \in H^2(\Omega) : v|_{\partial\Omega} = \frac{\partial v}{\partial\gamma}|_{\partial\Omega} = 0\}$. Let $H^{-1}(\Omega)$ be the "negative space" with norm given by $\|\cdot\|_{-1}$. Define Hilbert space $\mathbf{H} = H_0^2(\Omega) \times L^2(\Omega)$ with norm $\|(v, z)\|_{\mathbf{H}} = \|v\|_2 + \|z\|_0$, and define $\mathbf{H}_1 = H_0^1(\Omega) \times H^{-1}(\Omega)$ with norm $\|(v, z)\|_{\mathbf{H}_1} = \|v\|_1 + \|z\|_{-1}$.

In this paper, we suppose that the index of refraction $n(x) \in W^{1,\infty}(\Omega)$ and $1 + \delta \le n(x)$ in Ω for some constant $\delta > 0$; and the method and the theoretical analysis in this paper also hold for $n(x)$ strictly less than 1 in Ω.

Let $u = w - \sigma \in H_0^2(\Omega)$, $\omega = k^2 u$, $\lambda = k^2$, then from [16] we know that (2.1)–(2.4) can be rewritten as: Find $\lambda \in \mathbb{C}$, $(u, \omega) \in \mathbf{H} \setminus \{0\}$ such that

$$A((u, \omega), (v, z)) = \lambda B((u, \omega), (v, z)), \quad \forall (v, z) \in \mathbf{H}, \tag{2.5}$$

where

$$A((u, \omega), (v, z)) = ((\frac{1}{n-1} - \mu)\Delta u, \Delta v)_0 + \mu \int_\Omega \sum_{1\le i,j\le d} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \overline{v}}{\partial x_i \partial x_j} dx + (\omega, z)_0,$$

$$B((u, \omega), (v, z)) = (\nabla(\frac{1}{n-1} u), \nabla v)_0 + (\nabla u, \nabla(\frac{n}{n-1} v))_0 - (\frac{n}{n-1}\omega, v)_0 + (u, z)_0,$$

and $\mu > 0$ is chosen as good approximations of $\min(\frac{1}{n-1})$ such that $\frac{1}{n-1} - \mu \ge 0$.

From [16] we know that $A(\cdot, \cdot)$ is a selfadjoint and continuous sesquilinear form on $\mathbf{H} \times \mathbf{H}$, and

$$A((v, z), (v, z)) \ge \mu |v|_2^2 + \|z\|_0^2 \gtrsim \|(v, z)\|_{\mathbf{H}}^2, \quad \forall (v, z) \in \mathbf{H}. \tag{2.6}$$

We use $A(\cdot, \cdot)$ and $\|\cdot\|_A = \sqrt{A(\cdot, \cdot)}$ as the inner product and the norm on $\mathbf{H}$, respectively. Also from [16] we know $B((f, g), (v, z))$ is a continuous linear form on $\mathbf{H}$:

$$|B((f, g), (v, z))| \lesssim \|(f, g)\|_{\mathbf{H}_1} \|(v, z)\|_{\mathbf{H}_1}, \quad \forall (f, g), (v, z) \in \mathbf{H}_1. \tag{2.7}$$

The source problem associated with (2.5) is given by the following: Find $(\psi, \varphi) \in \mathbf{H}$ such that

$$A((\psi, \varphi), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}. \tag{2.8}$$

From the Lax–Milgram theorem we know that (2.8) has a unique solution. Therefore, we define the corresponding solution operators $T : \mathbf{H}_1 \to \mathbf{H}$ by

$$A(T(f, g), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}. \tag{2.9}$$

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