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# A variable exponent nonlocal $p(x)$-Laplacian equation for image restoration

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ABSTRACT

In this paper, we focus on the mathematical and numerical study of a variable exponent nonlocal $p(X)$-Laplacian equation for image denoising. Based on the Semigroup Theory, we prove the existence and uniqueness of solution for the proposed model. To illustrate the efficiency and effectiveness of our model, we provide the denoising experimental results as well we compare it with some existing models in the literature.



## 1. Introduction

Image processing is a set of techniques that convert an image into digital form by applying some operations on it, so as to extract some useful information from it or make it enhanced. The most important image processing task is image denoising that aims to seek the restored image $u(i)$ such that

$$f(i) = u(i) + \eta(i),$$

where $f(i)$ is a noisy image corrupted by the noise perturbation $\eta(i)$ at a pixel $i$ which often considered the stationary Gaussian with zero mean and variance $\sigma^2$. An efficient image denoising model is characterized by removing undesirable noise while preserving edges, details and textures. To handle this problem, many models were proposed such as bilateral filtering [1], wavelet thresholding [2] and variational models [3–6]. In [6] Rudin, Osher and Fatemi (ROF) proposed a very popular model for nonlinear image denoising. We recall that the ROF model regards $u$ as the solution to the following minimization problem

$$\min_u \left\{ E(u) = \int_\Omega |\nabla u| dx + \frac{\lambda}{2} \int_\Omega (f-u)^2 dx \right\},$$

where $\Omega \subset \mathbb{R}^N$ ($N$ is equal to 2 or 3 in practical situations) is an open and bounded domain with smooth boundary $\Gamma$, $f$ is a degraded image, $u$ is the restored image and $\lambda$ is a fidelity parameter. This method is well known for its edge preserving properties, but the experimental analysis readily showed the drawback named as staircasing effect. To overcome this, motivated by Chambolle and Lions regularization [7] which is a combination of isotropic diffusion in homogeneous regions and TV-based diffusion in edges, several approaches [4,6,8,9] have been proposed to approximate the energy $E(u)$ as

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follows:

$$\min_u \left\{ E_p(u) = \int_\Omega \Psi_p(x,\ |\nabla u|)dx + \frac{\lambda}{2}\int_\Omega (f-u)^2 dx \right\},$$

where $\Psi_p : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function, $\Psi_p(x,\ |\nabla u|) \sim |\nabla u|$ at edges where $|\nabla u|$ is large and the function $\Psi_p$ may depend on the location $x$ in the image [4,8,9]. The previous model exploits local image properties and performs well in removing noise. However, smaller details, such as textures, are destroyed. To overcome this problem, recently some kind of nonlocal methods [10–13] have been developed for image denoising which are able to remove noise, preserve edges and take care of the fine structures, details and textures. This new process of denoising models is based on the work of Yaroslavsky filter [14], and was introduced by the nonlocal means (Buades et al. and cf. [15]). The nonlocal methods use the similarities between neighboring or overlapping patches in an image. Gilboa and Osher [10,11] later formalized a systematical study for nonlocal image processing by introducing nonlocal operators. Furthermore, Kindermann, Osher and Jones [16,17] have proposed the nonlocal $p$-Laplacian problems for deblurring and denoising images, which can be written as:

$$\min_u \left\{ \mathcal{E}_p(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(x)-u(y)|^p dxdy + \frac{\lambda}{2}\int_\Omega (f-u)^2 dx \right\}.$$

The previous minimization is equivalent to the nonlocal analogous energy functional associated to the $E_p(u)$ with $\Psi_p(x,\ t) = \frac{t^p}{p}$. The images resulting from the application of the nonlocal $p$-Laplacian model are better enhanced, denoised, without staircasing effect and preserve small details and texture as compared to those of the model $E_p(u)$. However, the only drawback is that it takes a lot of iterations to obtain the denoised image. To remedy this, the idea is to use the benefits of combining the variable exponent that yields to use different diffusion types depending on each pixel in the image with the nonlocal $p$-Laplacian equation.

Inspired by the mentioned works, we propose a new nonlocal image denoising approach based on the variable exponent $p(X)$-Laplacian. The combination of the nonlocal $p$-Laplacian equation with the variable exponent allows a powerful and faster denoising process. The proposed model inherits the power of the nonlocal approach in preserving textures and small details, furthermore, the variable exponent helps the algorithm to converge quickly toward the solution. The proposed model can be written as

$$(P)\begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x,y)|u(t,y)-u(t,x)|^{p(x,y)-2}(u(t,y)-u(t,x))dy - \lambda(u(x)-f(x)) & \text{in } Q, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where the kernel $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative symmetric continuous smooth function with compact support contained in $\Omega \times B(0,d) \subset \mathbb{R}^N \times \mathbb{R}^N$ such that

$$0 < \sup_{y\in B(0,d)} J(x,y) = R(x) \in L^\infty(\Omega) \text{ and } \int_{\mathbb{R}^N} J(x,y)dx = 1 \tag{1}$$

and $p : \Omega \times \Omega \to (1,\infty)$ is a continuous symmetric variable exponent in $\Xi = \Omega \times \Omega$ such that

$$p(x,y) = p(y,x), \quad \underline{p}(x) = \inf_{y\in\bar{\Omega}} p(x,y), \quad \overline{p}(x) = \sup_{y\in\bar{\Omega}} p(x,y),$$
$$\underline{p}(x) \subset (\underline{p}^-, \underline{p}^+) \subset (1,+\infty) \quad \text{and} \quad \overline{p}(x) \subset (\overline{p}^-, \overline{p}^+) \subset (1,+\infty), \tag{2}$$

with finite constants $\underline{p}^\pm > 1$ and $\overline{p}^\pm > 1$.

The rest of this paper is structured as follows. In the next section, we present some basic definitions and preliminaries needed to state the results. In Section 2, we prove the existence and uniqueness of the solution for the problem $(P)$. At last, Section 3 is devoted to numerical results and comparative experiments to prove the efficiency of our model.

## 2. Preliminaries

Let $\Xi \subset \mathbb{R}^N$ be a bounded domain, $x \in \Xi$, $\partial\Xi$ be Lipschitz-continuous. Firstly, we recall the definitions of the function spaces $L^{p(x)}(\Xi)$ needed to prove the existence of the proposed model (for more instance see [18–21]). The exponents $q(x)$ are continuous in $\Xi$ such that

$$q(x) \subset (q^-, q^+) \subset (1,+\infty),$$

with finite constants $q^\pm > 1$. We define the *variable exponent Lebesgue* $L^{q(x)}(\Xi)$ to consist of all measurable functions $f$ on $\Xi$ such that

$$\varrho_{q(.)}(f) = \int_\Xi |f|^{q(x)}dx < \infty.$$

The function $\varrho_{q(.)} : \Xi \to [0,+\infty)$ is called *the modular* of the space $L^{q(.)}(\Xi)$ which is equipped with the norm

$$\|u\|_{q(.),\Xi} = \|u\|_{L^{q(x)}(\Xi)} = \inf\{\alpha > 0 : \varrho_{q(.)}(\frac{u}{\alpha}) \le 1\}.$$

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