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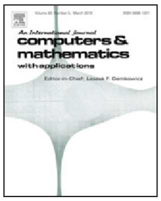

High-quality 2D mesh generation without obtuse and small angles

Dawar Khan [a,b], Dong-Ming Yan [a,*], Yiqun Wang [a,b], Kaimo Hu [c], Juntao Ye [a], Xiaopeng Zhang [a]

[a] *National Laboratory of Pattern Recognition, Institute of Automation, Chinese Academy of Sciences, Beijing, 100190, China*
[b] *University of Chinese Academy of Sciences, Beijing, 100049, China*
[c] *HPCG lab, Purdue University, West Lafayette, IN 47906, USA*

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ABSTRACT

In this paper, we present an efficient method to eliminate the obtuse triangles for high quality 2D mesh generation. Given an initialization (e.g., from Centroidal Voronoi Tessellation—CVT), a limited number of point insertions and removals are performed to eliminate obtuse or small angle triangles. A mesh smoothing and optimization step is then applied. These steps are repeated till a desired good quality mesh is reached. We tested our algorithm on various 2D polygonal domains and verified that our algorithm always converges after inserting a few number of new points, and generates high quality triangulation with no obtuse triangles.

1. Introduction

A mesh is a discretization of a geometric domain into small simple elements [1], where in 2D, triangles and quadrilaterals are most commonly used as the basic units. Since most objects are ultimately converted to meshes for efficient rendering and numerical solution of partial differential equations, mesh generation becomes one of the essential steps for most of the geometry processing applications. Furthermore, by mesh generation, sources that are represented as a variety of forms such as B-rep (Boundary representation), NURBS (Non-Uniform Rational Basis Spline) and point clouds can be processed in a unified way. In finite element mesh generation, the existence of small and/or large angle(s) badly affects the simulation results. Although there exists a huge amount of work for high-quality 2D mesh generation, there is still no practical algorithm for meshing a 2D domain without introducing obtuse triangles, except for a few theoretical approaches in literature.

For numerical stability of simulations, many methods have been proposed to eliminate the triangles with small angles that usually destroy the numerical conditioning [2–4]. However, obtuse and right triangles are also not preferred in many applications [5]. To the best of our knowledge, only one paper directly addresses the problem of acute triangulation [6]. This method iteratively minimizes its energy function (Eq. (1)) to ensure the well-centeredness of the triangulations in arbitrary dimensions

$$f_n(\sigma^n) = \max_{\text{vertices } v \in \sigma^n} \left| \frac{h(v - \sigma^n)}{R(\sigma^n)} - k_n \right|, \tag{1}$$

where $h(v - \sigma^n)$ represents the distance between $c(\sigma^n)$ and $c(\sigma_i^{n-1})$ (the circumcenters of the facets), $R(\sigma^n)$ is the circumradius of the facet and $k_n \in (0 \;\; 1]$ is a constant which might be dependent on dimension n of the n-simplex

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[*] Corresponding author.
E-mail address: dongming.yan@nlpr.ia.ac.cn (D.-M. Yan).

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σ^n. In order to ensure the well-centeredness of a mesh, the condition $h(v - \sigma^n) > 0$ must be valid for each vertex $v_i \in \sigma^n$. The experimental results reveal that the maximal angle has minimized well, however there still exist small angles (i.e. $angle < 30°$). Furthermore, the standard deviations of the angles recorded are also higher [6].

Although acute triangulation is not strictly required for some applications, such as covolume method [7], *Discrete Exterior Calculus* (DEC) [8] and space–time meshing [9], the computations would be much easier if such meshes were available [6].

Typical meshing algorithms do not guarantee acute triangulation. For instance, a Delaunay triangulation may contain obtuse angles if the given point sets are not well distributed. Even when the powerful *Centroidal Voronoi Tessellation* (CVT) optimization is applied, the results might still contain obtuse angles, especially on domain boundaries [10]. Although the method proposed [6] improves maximal angle, it usually introduces triangles with small angles, which might again destroy the numerical conditioning.

In this paper, we present a practical method for acute triangulation, while keeping all the triangles well shaped. Given an initial mesh with well-distributed vertices (e.g., generated by CVT), our method first defines local clusters that contain obtuse or right angles, and then modifies the local clusters by inserting new points and performing mesh smoothing and optimization. In addition to our own vertex relocation strategy the mesh smoothing is also aimed by CVT [11]. Consecutively, bad vertices are also removed from mesh to eliminates bad triangles (i.e. triangles with an $angle > 90°$ or $<30°$). Since the optimization step may again generate non-acute triangles, we iterate the above process until no bad triangles are introduced any more. In initial (200–300) iterations, we apply these operations globally throughout the mesh, while in later iterations, operations are applied locally in the neighborhood of bad triangles, leaving good triangles unaffected. Experiments show that our method does not only guarantee the acute triangulation, but also provides better overall triangle shapes compared with the state-of-the-art approaches.

In general, we claim the following contributions.

- A novel pipeline for acute triangulation, which at the same time keeps all the triangles well shaped.
- An empirical validation to show that our method can guarantee the acute triangulation.
- A complete comparison with the state-of-the-art methods to show the advantages of our method.

2. Related work

Algorithms for generating high quality mesh can be divided into constructive and iterative approaches. Usually, the constructive approaches begin with only the input boundaries. Then, the desired triangulation is generated by adding additional Steiner points inside the region or on the boundary. On the contrary, the iterative approaches begin with an existing triangulation, where Steiner points are then added/removed/relocated iteratively to generate a high quality mesh.

As for iterative methods, the most intuitive way to improve the mesh quality is smoothing, which relocates the positions of vertices while keeping their connections. Renka proposed a mesh smoothing method that consists of minimizing the sum of squared element volumes over the free vertex positions [12], and furthermore, they presented a new angle-based method to improve the mesh quality. Chen developed several mesh smoothing schemes using Optimal Delaunay Triangulation (ODT) as a framework [13], which can also be applied in anisotropic case. Smoothing improves the mesh quality dramatically in a simple way. However, since it does not modify the local connections, if the initial triangulation is not regular enough, the results will be poor.

More powerful approaches are based on the combination of constructive methods and iterative methods. By modifying both the connections and positions of the vertices, many proposed approaches can guarantee the minimal and/or maximal angles in the result triangulation. Based on maximal Poisson disk sampling, Ebeida et al. [14] presented a Conforming Delaunay Triangulation (CDT) algorithm, which can make the unconstrained angles in the results between 30° and 120°. The same angle range can be empirically guaranteed in [15]. Yan and Wonka [16] studied the generation of maximal Poisson-disk sets with varying radii, and applied the according algorithm to surface remeshing, which empirically makes the triangle angles between 32° and 120°. Chew presented an efficient technique based on Delaunay triangulation [17], which guarantees the angles in the resulting triangles are all between 30° and 120°. Sieger et al. [18] noticed that short edges in the Voronoi diagrams are harmful for simulation with polygonal meshes. To solve this issue, they proposed an energy function that minimizes the sum of squared distances from a triangle's circumcenter to its inscribed center [18]. Tournois et al. [19] addressed the problem of generating 2D quality triangle meshes from a set of constraints provided as a planar straight line graph. Their method interleaves Delaunay refinement and optimization, and the practical bounds of the results are usually between 30° and 100°. Although the above methods avoid too low and/or too high angles empirically or theoretically, the results usually contain obtuse triangles.

Since non-obtuse triangulation is preferred in many applications, such as covolume method [7], Discrete Exterior Calculus [8] and space–time meshing [9], some non-obtuse triangulation approaches have been proposed. In 2007, Erten and Üngör proposed the first module for computing acute triangulation in two dimensional domains [20]. Two years later, they proposed another method which not only maximizes small angles but also minimizes large angles [21]. Their first method [20] usually introduces needle-shaped angles. The second method [21], uses a given minimal angle constraint α and a maximum angle constraint γ, to compute a triangulation of the domain such that all angles are in $[\alpha, \gamma]$. Experiments show that α can be as high as 41°, while γ can be as low as 81°. However, they cannot guarantee all the inputs are triangulated in acute angles.

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