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A stabilized finite volume method for the evolutionary Stokes–Darcy system

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ABSTRACT

In this paper, we propose a stabilized fully discrete finite volume method based on two local Gauss integrals for a non-stationary Stokes–Darcy problem. This stabilized method is free of stabilized parameters and uses the lowest equal-order finite element triples $P1$–$P1$–$P1$ for approximating the velocity, pressure and hydraulic head of the Stokes–Darcy model. Under a modest time step restriction in relation to physical parameters, we give the stability analysis and the error estimates for the stabilized finite volume scheme by means of a relationship between finite volume and finite element approximations with the lower order elements. Finally, a series of numerical experiments are provided to demonstrate the validity of the theoretical results.

1. Introduction

Finite difference, finite element and finite volume methods are the three most important classes of numerical methods for partial differential equations. Most recently, the finite volume method has received more and more concerns because it enjoys not only the simplicity of finite difference method but also the accuracy of finite element method [1–4]. It has some advantages as follows: Firstly, it inherits the conservation law, which is fairly desirable; Secondly, the computational effort is greater than finite difference method and less than finite element method, while the finite volume method has the same accuracy as the finite element method; Thirdly, it could deal with complicated geometries and its accuracy is higher than the finite difference methods [2]. The finite volume method is also called a generalized difference method or a control volume method. It has been widely utilized to solve many types of partial differential equations, such as, second order elliptic equations, the Stokes equations, the non-stationary conduction–convention problem [5–10].

In recent decades, there have been rising concerns about multi-modeling problems in real world applications. In particular, a lot of numerical methods have been carried out for the coupling Stokes–Darcy model, such as finite element methods [11–14], discontinuous Galerkin methods [15–18], mortar discretization [19,20] and boundary integral methods [21,22]. However, all of these works make use of the finite element methods for the spatial discretization to solve the coupled Stokes–Darcy system. In this paper, a parameter-free stabilized finite volume method based on two local Gauss integrals is developed for the non-stationary Stokes–Darcy problem. We make use of the lowest equal-order finite element triples $P1$–$P1$–$P1$ for the approximation of velocity, pressure, and hydraulic head. In practical application, the lower order finite elements, such as $P1$ and $P0$ elements, tend to get more attention of engineers. The lower order finite elements are convenient and efficient, what's more, they can provide enough accuracy. However, both $P1$–$P1$ and $P1$–$P0$ are not stable for Stokes equations, so we add a suitable stabilization term in the discrete formulation so that the $P1$–$P1$ pair for velocity and pressure in Stokes equation satisfies the inf–sup condition. The stabilization technique based on the residual of a local Gauss

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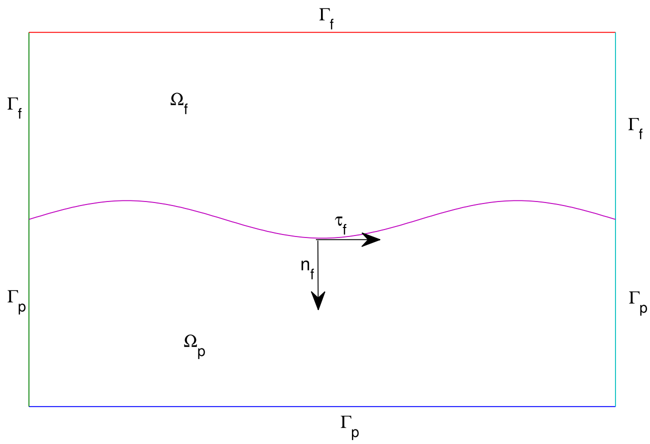

Fig. 1. A global domain Ω consisting of the fluid region Ω_f and the porous media region Ω_p separated by the interface Γ.

integration term on each element level is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and can be implemented at the element level. Many other stabilization techniques have been widely studied in [23–26]. Moreover, under a modest time step restriction in relation to physical parameters, we prove almost unconditional stability (without a time step restriction linked to the spacial mesh width) of the stabilized finite volume method and obtain the error estimates between the solution of the stabilized finite volume element approximation and the exact solution.

The paper is organized as follows. In Section 2, we present the non-stationary Stokes–Darcy model and the weak formulation. The fully discrete stabilized finite volume formulation for the non-stationary Stokes–Darcy is proposed in Section 3. In Section 4, the stability analysis is provided for the new method. Section 5 is devoted to the convergence analysis for the stabilized finite volume scheme. In Section 6, numerical tests are given to illustrate the accuracy and efficiency of the stabilized method. Finally, the conclusions are drawn in Section 7.

2. The Stokes–Darcy model and variational formulation

We consider a coupled Stokes–Darcy model in a bounded domain $\Omega \in R^2$, consisting of a fluid region Ω_f and a porous medium region Ω_p, with interface $\Gamma = \partial\Omega_f \cap \partial\Omega_p$. Both Ω_f and Ω_p have Lipschitz continuous boundaries. Define $\Gamma_i = \partial\Omega_i \setminus \Gamma$ for $i = f,\ p$. Moreover, we denote by $\overline{\Omega} = \overline{\Omega_f} \cup \overline{\Omega_p}$, n_f and n_p the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, and τ_f the unit tangential vector on the interface Γ. Note that $n_p = -n_f$ on Γ (see Fig. 1).

In the free flow region Ω_f, the fluid flow is governed by the Stokes equations:

$$\frac{\partial u}{\partial t} - \nu\triangle u + \nabla p = f_1 \quad in\ \Omega_f, \tag{2.1}$$

$$\nabla\cdot u = 0 \quad in\ \Omega_f. \tag{2.2}$$

Here, ν is the kinetic viscosity, u denotes the fluid velocity, p denotes the kinematic pressure, and f_1 denotes a general body force that includes gravitational acceleration.

In the porous medium region Ω_p, the flow is governed by the Darcy equations:

$$S_0\frac{\partial \phi}{\partial t} + \nabla\cdot u_p = f_2 \quad in\ \Omega_p, \tag{2.3}$$

$$u_p = -\mathbf{K}\nabla\phi \quad in\ \Omega_p. \tag{2.4}$$

where u_p is the specific discharge rate in the porous medium, $\mathbf{K}$ is the hydraulic conductivity tensor, f_2 is a source term, S_0 is the soil compressibility, and ϕ denotes the hydraulic head, which is defined as $\phi = z + \frac{p_p}{\rho g}$. Here, p_p is the dynamic pressure, ρ is the density, z is the relative depth from an arbitrary fixed reference level, and g is the gravitational acceleration. Without loss of generality, we assume $z = 0$ and $\mathbf{K}$ is a symmetric and positive definite matrix and there exist $K_{min}, K_{max} > 0$ such that $K_{min}|x|^2 \le \mathbf{K}x\cdot x \le K_{max}|x|^2$, $a.e. x \in \Omega_p$. Combining the continuity equation (2.3) with *Darcy's* law (2.4), we can have the following equation

$$S_0\frac{\partial \phi}{\partial t} - \nabla\cdot(\mathbf{K}\nabla\phi) = f_2 \quad in\ \Omega_p. \tag{2.5}$$

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