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Numerical analysis and testing of a stable and convergent finite element scheme for approximate deconvolution turbulence models

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ABSTRACT

This report presents a stable and convergent finite element scheme for the approximate deconvolution turbulence models (ADM). The ADM is a popular turbulence model intensely studied lately but the computation of its numerical solution raises issues in terms of efficiency and accuracy. This report addresses this question. The proposed scheme presented herein is based on a new interpretation of the ADM model recently introduced by the author. Following this interpretation, the solution of the ADM is viewed as the average of a perturbed Navier–Stokes system. The scheme uses the Crank–Nicolson time discretization and the finite element spatial discretization and is proved to be stable and convergent provided a moderate choice of the time step is made. Numerical tests to verify the convergence rates and performance on a benchmark problem are also provided and they prove the correctness of this approach to numerically solve the ADM.

1. Introduction

This report presents a stable and convergent scheme to solve the approximate deconvolution turbulence models (ADM for short) numerically. The ADM have been introduced by Stolz, Adams and Kleiser in [1–3] and later studied and tested in many papers such as [4–14], see also Layton and Rebholz, [15]. Their study and testing is motivated by the fact that the simulation of turbulent flows based on the Navier–Stokes equations is not efficient computationally due to the large number of degrees of freedom required by such simulations, [16]. The ADM aim at approximating averages of flow velocities, to allow such simulations with much less degrees of freedom. The ADM is obtained from the Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} &= \mathbf{f}\,, \quad \text{in } \Omega\times(0,T) \\ \nabla\cdot\mathbf{u} &= 0\,, \quad \text{in } \Omega\times(0,T) \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned}$$

to which a filtering operator $\mathbf{w} \to \overline{\mathbf{w}}$ is applied, and under the assumption that filtering and differentiation will commute (which is mathematically correct only in special conditions, such as periodic boundaries, see [17]) we obtain

$$\overline{\mathbf{u}}_t + \overline{\mathbf{u}\cdot\nabla\mathbf{u}} - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}}.$$

To obtain the ADM, one may use a deconvolution operator D that has the property $\mathbf{u} \approx D\overline{\mathbf{u}}$ (any method to solve the ill posed problem of recovering $\mathbf{u}$ from $\overline{\mathbf{u}}$ would lead to such an operator) to write then

$$\overline{\mathbf{u}}_t + \overline{D\overline{\mathbf{u}}\cdot\nabla D\overline{\mathbf{u}}} - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} \approx \overline{\mathbf{f}}.$$

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Such approximate deconvolution operators are the van Cittert deconvolution operators, [18], or the multiscale deconvolution operators, [19]. The ADM is then given by

$$\mathbf{W}_t + \overline{D\mathbf{W} \cdot \nabla D\mathbf{W}} - \nu \Delta \mathbf{W} + \nabla q = \overline{\mathbf{f}} \tag{1}$$

and its solution $\mathbf{W}$ is the approximation of average velocities $\overline{\mathbf{u}}$ whereas q is a pressure approximation. In most studies listed above, the filtering operator that has been used is the Helmholtz filter (or the differential filter) [20]

$$G\mathbf{u} = \overline{\mathbf{u}} = (I - \alpha^2 \Delta)^{-1} \mathbf{u},$$

which is used in many other turbulence models as well, such as the Leray-α, modified Leray-α or NS-α models, [4–6,21–27].

This paper is concerned with the numerics of the ADM. Numerical schemes for the ADM model (1) have been proposed and studied in [11,14,13,12].

One issue that has to be solved is that in Eq. (1) the nonlinear term

$$\overline{D\mathbf{W} \cdot \nabla D\mathbf{W}}$$

in the discrete setting requires filtering a quantity ($D\mathbf{W} \cdot \nabla D\mathbf{W}$) whose value on the boundary is not generally known (i.e. $\nabla D\mathbf{W}$ is not known). Therefore the nonlinearity should by unfiltered in the numerical scheme. This is generally done by choosing the test function $(I - \alpha^2 \Delta)\mathbf{v}$ (instead of $\mathbf{v}$) but this raises other issues (it increases the order of the resulting system).

In [11] the scheme is valid for $N = 0$ and the problem of filtering the nonlinear term is solved by casting the model into a fourth order formulation. The paper [14] considers the two dimensional case, the model is again cast into a fourth order formulation and the nonlinear term is treated explicitly (is lagged in the previous iteration). Up to now, it seems that the most mathematically sound scheme for ADM is the reduced ADM algorithm presented and investigated by Rebholz and his collaborators in [12,13], where the test function in (1) is set as $(I - \alpha^2 \Delta)\mathbf{v}$ leading to the variational formulation

$$(\mathbf{W}_t, \mathbf{v}) + \alpha^2 (\nabla \mathbf{W}_t, \nabla \mathbf{v}) + (D\mathbf{W} \nabla \cdot D\mathbf{W}, \mathbf{v}) \\ + \nu (\nabla \mathbf{W}, \nabla (I - \alpha^2 \Delta)\mathbf{v}) + (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

which, under the approximation assumption $(I - \alpha^2 \Delta)\mathbf{v} \approx D\mathbf{v}$, will become the reduced ADM model studied in [12,13]:

$$(\mathbf{W}_t, \mathbf{v}) + \alpha^2 (\nabla \mathbf{W}_t, \nabla \mathbf{v}) + (D\mathbf{W} \nabla \cdot D\mathbf{W}, \mathbf{v}) \\ + \nu (\Delta \mathbf{W}, \nabla D\mathbf{v}) + (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}).$$

For this modified ADM the nonlinearity does not have to be filtered and for the case $D = D_N$ with $N = 1$ (the first order van Cittert deconvolution model) the authors proved the model is almost unconditionally stable (it requires a time step restriction) and optimally convergent. In this case the averages $\overline{\mathbf{W}}$ that appear in $D\mathbf{W}$ are lagged in the previous iteration and also, in the numerical scheme the resulting nonlinearity is linearized, therefore each time step requires only a linear solve making the scheme very efficient computationally.

In this report it is presented a numerical scheme for the unmodified ADM. The scheme is based on a new interpretation of the ADM which has been recently presented in [28]. Therein, the exact solution $\mathbf{W}$ of the ADM (1) is proved to be precisely the average $\mathbf{W} = \overline{\mathbf{w}}$ of the solution $\mathbf{w}$ of the model

$$\mathbf{w}_t + \nabla \cdot D\overline{\mathbf{w}} \otimes D\overline{\mathbf{w}} - \nu \Delta \mathbf{w} + \nabla p = \mathbf{f}, \tag{2}$$

since upon filtering the above equation, it will become exactly the ADM with solution $\overline{\mathbf{w}}$.

The above model (2), which is a perturbation of the NSE, is solved (for $\mathbf{w}$, which is an approximation of the NSE solution $\mathbf{u}$) using the Crank–Nicolson scheme and then filtered to get $\overline{\mathbf{w}}$ which will be the ADM approximation of the filtered NSE solution $\overline{\mathbf{u}}$. In the sequel the model (2) will be quoted as the ADM model. Our numerical scheme uses the van Cittert approximate deconvolution operators D_N. The FEM analysis of the model (2) resembles the techniques used in [29] for the Leray-deconvolution model and in [27] for the Navier–Stokes-α model. Numerical tests to confirm the predicted rates are provided in the last section. A test on a benchmark problem, channel flow with two outlets and a contraction, is also presented. Both show good results and motivate further investigation and testing of the presented scheme.

2. Mathematical context

This section presents several notations and standard concepts and properties that will be used in the analysis. $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, will denote a regular, bounded, polyhedral domain and we let $\|\cdot\|$ and $(\cdot, \cdot)$ denote the usual $L^2(\Omega)$ norm and inner product. $\|\cdot\|_{L^p}$ denotes the usual norm on the space $L^p(\Omega)$. Sobolev space are denoted by $W_p^k(\Omega)$ with $\|\cdot\|_k$ norm and $|\cdot|_{W_p^k}$ standard semi-norm . H^k will denote the Sobolev space W_2^k with norm $\|\cdot\|_k$. For vector valued functions the norms are denoted by

$$\|v\|_{L^\infty(0,T;H^k)} := \sup_{0<t<T} \|v(t,\cdot)\|_k , \quad \|v\|_{L^p(0,T;H^k)} := \left(\int_0^T \|v(t,\cdot)\|_k^p \, dt \right)^{1/p} .$$

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