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The Uzawa-PPS iteration methods for nonsingular and singular non-Hermitian saddle point problems ☆

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ABSTRACT

In this paper, based on the positive-definite and positive-semidefinite splitting (PPS) iteration scheme, we establish a class of Uzawa-PPS iteration methods for solving nonsingular and singular non-Hermitian saddle point problems with the (1,1) part of the coefficient matrix being non-Hermitian positive definite. Theoretical analyses show that the convergence and semi-convergence properties of the proposed methods can be guaranteed under suitable conditions. Furthermore, we consider acceleration of the Uzawa-PPS methods by Krylov subspace (like GMRES) methods and discuss the spectral properties of the corresponding preconditioned matrix. Numerical experiments are given to confirm the theoretical results which show that the feasibility and effectiveness of the proposed methods and preconditioners.

1. Introduction

In this paper, we consider the following large and sparse saddle point problems of the form

$$\mathcal{A}u \equiv \begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$ is a non-Hermitian positive definite matrix, i.e., the Hermitian part of A is positive definite, $B \in \mathbb{C}^{m\times n}$ is a rectangular matrix, $p \in \mathbb{C}^m$ and $q \in \mathbb{C}^n$ are given vectors with $m \geqslant n$, B^* denotes the transpose conjugate of B. It follows that the linear system (1.1) is nonsingular when matrix B is of full column rank and singular when matrix B is rank deficient [1]. This kind of linear system (1.1) frequently arises in many different applications of scientific computing and engineering applications, such as mixed finite element approximation to solve the Stokes and Navier–Stokes equations, computational fluid dynamics, constrained optimization, optimal control and so on; see [1–5] and references therein.

On account of the matrices A and B are usually large and sparse, in practical issues, for this reason iteration methods become more efficient in storage requirements and preservation sparsity even more attractive than direct methods for the linear system (1.1). When matrix B is of full column rank, i.e., the linear system (1.1) is nonsingular, a number of efficient iteration methods have been proposed in many literatures, including Uzawa-type methods [6–10], SOR-like methods [11–14], HSS-type methods [15–19], RPCG iteration methods [20,21] and Krylov subspace iteration methods [22,23]. When matrix B is rank deficient, i.e., the linear system (1.1) is singular, there are also many efficient iteration methods for solving

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this case, such as Uzawa-type methods [24,25], Krylov subspace iteration methods [23,26,27] and matrix splitting iteration methods [28–30].

Among these methods, the classic Uzawa method [31] received wide attention and obtained considerable achievements in recent years. The iteration scheme is given by

$$\begin{cases} x^{(k+1)} = A^{-1}(p - By^{(k)}) \\ y^{(k+1)} = y^{(k)} + \tau(B^* x^{(k+1)} - q) \end{cases} \tag{1.2}$$

where τ is a positive parameter. Note that the linear subsystem $Ax = f$ need to be solved at each step of Uzawa method, it is suitable to use iteration method to approximate its solution since matrix A is always large and sparse. For example, when matrix A is Hermitian positive definite, Zhang et al. in [32] introduced a class of Uzawa-SOR iteration methods by utilizing one-step successive over relaxation (SOR) iteration to approximate $x^{(k+1)}$ in each step of Uzawa method. Yun in [33] further studied three variants of Uzawa methods and obtained corresponding convergence results. Moreover, when matrix A is non-Hermitian positive definite, due to the performance and elegant mathematical properties of HSS methods, Yang et al. in [34] and Liu et al. in [35] established a class of Uzawa-HSS (or NHSS-like) iteration methods by using one-step Hermitian and skew-Hermitian splitting (HSS [15]) iteration to approximate $x^{(k+1)}$ in each step of Uzawa method, the Uzawa-HSS iteration scheme can be described as

$$\begin{cases} (\alpha I + H)(\alpha I + S)x^{(k+1)} = (\alpha I - H)(\alpha I - S)x^{(k)} + 2\alpha(p - By^{(k)}) \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(B^* x^{(k+1)} - q) \end{cases} \tag{1.3}$$

where α and τ are two given positive constants, matrix Q is a given Hermitian positive definite, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ are the Hermitian part and the skew-Hermitian part of matrix A, respectively. Furthermore, Cao et al. in [36] further presented a class of Uzawa-PSS iteration methods by exploiting one-step positive-definite and skew-Hermitian splitting (PSS [18]) iteration to approximate $x^{(k+1)}$ in each step of Uzawa method. In addition, theoretical analysis and numerical experiments show that these methods are very feasible and effective for solving large and sparse nonsingular and singular saddle point problems (1.1), and see [37,38] for singular case.

In this paper, based on the positive-definite and positive-semidefinite splitting (PPS) iteration scheme proposed by Huang et al. [39] for solving non-Hermitian positive definite linear system, we establish a class of Uzawa-PPS iteration methods by utilizing one-step PPS iteration to approximate $x^{(k+1)}$ in each step of Uzawa method for solving nonsingular and singular non-Hermitian saddle point problems (1.1). The convergence and semi-convergence of the Uzawa-PPS methods are discussed. Moreover, the Uzawa-PPS methods can provide an efficient preconditioner to acceleration by Krylov subspace methods and the spectral properties of the corresponding preconditioned matrix are studied in detail. Finally, two experiments are given to confirm the theoretical results which shows that the feasibility and effectiveness of the proposed methods and preconditioners.

The rest of this paper is organized as follows. In Section 2, we introduce the Uzawa-PPS iteration methods for solving non-Hermitian saddle-point linear system (1.1). We study the convergence and semi-convergence analysis of the Uzawa-PPS methods under suitable conditions in Section 3. Moreover, we discuss the spectral properties of the corresponding preconditioned matrix in Section 4. Numerical results show the feasibility and effectiveness of the proposed methods and preconditioners in Section 5. Finally, we end this work with a brief conclusion in Section 6.

We adopt the following notation throughout this paper. I denotes the identity matrix with the appropriate dimension and 0 denotes the 0-matrix with the appropriate dimension. For a number a_0, the $\Re(a_0)$ and $\Im(a_0)$ denote the real and imaginary parts of a_0. For a vector x, the x^* and $\|x\|$ denote the conjugate transpose and the l_2 norm of the vector x, respectively. For a matrix H, the H^{-1}, H^*, $\mathcal{R}(H)$, $\mathcal{N}(H)$, $rank(H)$, $\sigma(H)$ and $\rho(H)$ denote the inverse, the conjugate transpose, the range spaces, the null spaces, the rank, the spectrum and the spectral radius of H, respectively.

2. The Uzawa-PPS iteration methods

In this section, we first establish the Uzawa-PPS iteration methods for solving the linear system (1.1). Based on the splitting techniques [39], the non-Hermitian positive definite matrix A can be split into

$$A = P_1 + P_2,$$

where P_1 and P_2 are positive definite and positive semidefinite matrices, respectively. Analogous to the classical alternating direction implicit iteration technique, we need to solve the first linear equation of the Uzawa method (1.2)

$$Ax^{(k+1)} = p - B^* y^{(k)}$$

by using the following PPS iteration scheme:

$$\begin{cases} (\alpha I + P_1)x^{(k+\frac{1}{2})} = (\alpha I - P_2)x^{(k)} + (p - By^{(k)}), \\ (\alpha I + P_2)x^{(k+1)} = (\alpha I - P_1)x^{(k+\frac{1}{2})} + (p - By^{(k)}), \end{cases} \tag{2.1}$$