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On multi-soliton solutions for the (2 + 1)-dimensional breaking soliton equation with variable coefficients in a graded-index waveguide

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ABSTRACT

In this work, we construct multi-soliton solutions of the (2 + 1)-dimensional breaking soliton equation with variable coefficients by using the generalized unified method. We employ this method to obtain double- and triple-soliton solutions. Furthermore, we study the nonlinear interactions between these solutions in a graded-index waveguide. The physical insight and the movement role of the waves are discussed and analyzed graphically for different choices of the arbitrary functions in the obtained solutions. The interactions between the solitons are elastic whether the coefficients of the equation are constants or variables.

1. Introduction

In recent years, attention has been paid to nonlinear evolution equations (NLEEs) in different branches of science such as biology, fluid mechanics, plasmas, condensed matter, and nonlinear optics [1–5]. Efforts have been dedicated to find the analytic solutions of NLEEs, including lump solutions (which could be obtained by using symbolic computations to many nonlinear wave equations including the Kadomtsev–Petviashvili equation) [6], solitonic and periodic rational solutions [7–9].

In order to study the solutions of NLEEs, some effective methods have been introduced such as a refined invariant subspace method [10,11] (which gives a largest possible solution subspace by choosing a specific solution to a linear ordinary differential equation as a basis solution), improved Hirota method, Hirota's bilinear method and its simplified form that were used to solve more NLEEs and could lead to establish multiple wave solutions, variable separation method, Darboux transformation, Bäcklund transformation, the inverse scattering method in which they contain different techniques for constructing these solutions, and the multiple exp-function method [12–25].

Hereby, we aim to apply the generalized unified method (GUM) [26–30] to find multi-soliton solutions of the (2 + 1)-dimensional breaking soliton equation with variable coefficients, denoted by 2D-vcBSE, and study these solutions in a graded-index waveguide.

The 2D-vcBSE is given by

$$\begin{aligned} u_t + \alpha(t)\,u_{xxy} + \gamma(t)\,(u\,v)_x = 0, \\ u_y = \beta(t)\,v_x, \end{aligned} \tag{1}$$

where $u = u(x, y, t),\ v = v(x, y, t)$ while $\alpha(t),\ \gamma(t)$ and $\beta(t)$ are arbitrary analytic functions.

The 2D-vcBSE is widely used when $\gamma(t) = 4\,\alpha(t)$ and $\beta(t) = 1$ which describe the (2 + 1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis [31,32]. In this case, Eq. (1) is solved by

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the projective Riccati equation expansion method [33] and the two general solutions are obtained for Eq. (1) by the singular manifold method [32]. Dai derived 2D-vcBSE chaotic behaviors by the mapping method [34].

The structure of this paper is as follows: In Section 2, the 2D-vcBSE is studied and its multi-soliton solutions are derived by using GUM. Also, through the graphic analysis in graded-index waveguide, the properties of these solutions will be analyzed and investigated. Section 3 is devoted to our conclusions.

2. Multi-soliton solutions of 2D-vcBSE by using GUM

In this section, we apply GUM (for details see [26–30]) to find multi-soliton solutions (double- and triple-soliton solutions) of 2D-vcBSE given by Eq. (1).

To obtain N-soliton solutions in the rational functions form we use, for instance, the transformations $u(x,y,t) = u_{1x}(x,y,t)$, $v(x,y,t) = v_{1x}(x,y,t)$ in Eq. (1). By using these transformations and integrating both sides with respect to x, Eq. (1) can be written as

$$\begin{aligned} &u_{1t} + \alpha(t)\,u_{1xxy} + \gamma(t)\,u_{1x}\,v_{1x} = 0,\\ &\qquad u_{1y} = \beta(t)\,v_{1x}, \end{aligned} \tag{2}$$

where the constants of integration are considered to be zero.

We mention that, in Eq. (2) by plugging u_{1y} from the second equation into the first equation and viewing v_1 as a part of the coefficients, we can transform Eq. (2) into a linear partial differential equation of first order, whose general solution can be systematically presented. So, if we take v_1 as a separable function (say $v_1 = a(t)\,b(x)$), then a general solution to the resulting linear equation could be given explicitly.

2.1. Double-soliton solutions of 2D-vcBSE by using GUM

Here, we use GUM to find two-soliton solutions of Eq. (2). Assume that

$$\begin{aligned} u_1(x,y,t) = U(\xi_1,\xi_2) &= \frac{p_0(t) + p_1(t)\,\phi_1(\xi_1) + p_2(t)\,\phi_2(\xi_2) + p_{12}(t)\,\phi_1(\xi_1)\,\phi_2(\xi_2)}{q_0(t) + q_1(t)\,\phi_1(\xi_1) + q_2(t)\,\phi_2(\xi_2) + q_{12}(t)\,\phi_1(\xi_1)\,\phi_2(\xi_2)},\\ v_1(x,y,t) = V(\xi_1,\xi_2) &= \frac{r_0(t) + r_1(t)\,\phi_1(\xi_1) + r_2(t)\,\phi_2(\xi_2) + r_{12}(t)\,\phi_1(\xi_1)\,\phi_2(\xi_2)}{q_0(t) + q_1(t)\,\phi_1(\xi_1) + q_2(t)\,\phi_2(\xi_2) + q_{12}(t)\,\phi_1(\xi_1)\,\phi_2(\xi_2)}, \end{aligned} \tag{3}$$

where $\xi_1 = \alpha_1 x + \alpha_2 y + \int \alpha_3(t)\,dt$, $\xi_2 = \beta_1 x + \beta_2 y + \int \beta_3(t)\,dt$ and $p_i(t)$, $q_i(t)$, $r_i(t)$, $p_{12}(t)$, $q_{12}(t)$, $r_{12}(t)$ are arbitrary functions, $i = 0, 1, 2$.

The auxiliary functions $\phi_j(\xi_j)$ satisfy the auxiliary equations $\phi_j'(\xi_j) = c_j(t)\,\phi_j(\xi_j)$, where $c_j(t)$ are arbitrary functions, $j = 1, 2$.

By substituting from (3) into (2) and by equating the coefficients of $\phi_j(\xi_j)$ to be zero, we get a set of algebraic equations. By using any package in symbolic computations (such as the elimination method or other suitable solvable method with the aid of MATHEMATICA or MAPLE), we get

$$\begin{aligned} &p_0(t) = \frac{A_-\,H_-\,p_1(t)\,q_2(t)}{A_+\,H_+\,q_{12}(t)} - \alpha_1\,c_1(t)\,q_0(t)\,\lambda(t),\;\; p_2(t) = q_2(t)\left(\frac{A_-\,H_-\,p_1(t)\,q_2(t)}{A_+\,H_+\,q_{12}(t)\,q_0(t)} - \lambda(t)\,H_-\right),\\ &\qquad p_{12}(t) = \frac{A_-\,H_-\,p_1(t)\,q_2(t)}{A_+\,H_+\,q_0(t)} + \beta_1\,c_2(t)\,q_{12}(t)\,\lambda(t),\quad q_1(t) = \frac{A_+\,H_+\,q_{12}(t)\,q_0(t)}{A_-\,H_-\,q_2(t)}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &r_1(t) = \frac{A_+\,H_+\,q_{12}(t)(6\,\alpha_2\,\alpha(t)\,c_1(t)\,q_0(t) + r_0(t)\,\gamma(t))}{A_-\,H_-\,q_2(t)\,\gamma(t)},\; r_2(t) = q_2(t)\left(\frac{r_0(t)\,\gamma(t) + 6\,\beta_2\,c_2(t)\,q_0(t)\,\alpha(t)}{q_0(t)\,\gamma(t)}\right),\\ &\qquad r_{12}(t) = \frac{q_{12}(t)(6\,\alpha_2\,c_1(t)\,q_0(t)\,\alpha(t) + 6\,\beta_2\,c_2(t)\,q_0(t)\,\alpha(t) + r_0(t)\,\gamma(t))}{q_0(t)\,\gamma(t)},\\ &\qquad \beta_3(t) = -\beta_1^2\,\beta_2\,c_2^2(t)\,\alpha(t),\quad \alpha_3(t) = -\alpha_1^2\,\alpha_2\,c_1^2(t)\,\alpha(t), \end{aligned} \tag{5}$$

where $A_\pm = \alpha_1(2\,\alpha_2\,\beta_1 + \alpha_1\,\beta_2)\,c_1(t) \pm \beta_1(\alpha_2\,\beta_1 + 2\,\alpha_1\,\beta_2)\,c_2(t)$, $H_\pm = \alpha_1\,c_1(t) \pm \beta_1\,c_2(t)$, and $\lambda(t) = \frac{6\,\alpha(t)\,\beta(t)}{\gamma(t)}$.

By solving the auxiliary equations $\phi_j'(\xi_j) = c_j(t)\,\phi_j(\xi_j)$, $j = 1, 2$ and substituting together with (4)–(5) into (3), we get the solution of Eq. (1) namely

$$\begin{aligned} &u(x,y,t) = u_{1x}(x,y,t),\;\; u_1(x,y,t) = U(\xi_1,\xi_2) =\\ &\frac{p_0(t) + p_1(t)\,e^{c_1(t)\,\xi_1} + q_2(t)\,(\frac{A_-\,H_-\,p_1(t)\,q_2(t)}{A_+\,H_+\,q_{12}(t)\,q_0(t)} - \lambda(t)\,H_-)e^{c_2(t)\,\xi_2}}{q_0(t)(1 + \frac{A_+\,H_+\,q_{12}(t)\,e^{c_1(t)\,\xi_1}}{A_-\,H_-\,q_2(t)}) + q_2(t)\,e^{c_2(t)\,\xi_2} + q_{12}(t)\,e^{c_1(t)\,\xi_1 + c_2(t)\,\xi_2}}\\ &+ \frac{(\frac{A_-\,H_-\,p_1(t)\,q_2(t)}{A_+\,H_+\,q_{12}(t)\,q_0(t)} + \beta_1\,c_2(t)\,q_{12}(t)\,\lambda(t))e^{c_1(t)\,\xi_1 + c_2(t)\,\xi_2}}{q_0(t)(1 + \frac{A_+\,H_+\,q_{12}(t)\,e^{c_1(t)\,\xi_1}}{A_-\,H_-\,q_2(t)}) + q_2(t)\,e^{c_2(t)\,\xi_2} + q_{12}(t)\,e^{c_1(t)\,\xi_1 + c_2(t)\,\xi_2}}, \end{aligned} \tag{6}$$