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Analysis of three-dimensional anisotropic heat conduction problems on thin domains using an advanced boundary element method

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ABSTRACT

In this paper, an advanced boundary element method (BEM) is developed for solving three-dimensional (3D) anisotropic heat conduction problems in thin-walled structures. The troublesome nearly singular integrals, which are crucial in the applications of the BEM to thin structures, are calculated efficiently by using a nonlinear coordinate transformation method. For the test problems studied, promising BEM results with only a small number of boundary elements have been obtained when the thickness of the structure is in the orders of micro-scales (10^{-6}), which is sufficient for modeling most thin-walled structures as used in, for example, smart materials and thin layered coating systems. The advantages, disadvantages as well as potential applications of the proposed method, as compared with the finite element method (FEM), are also discussed.

1. Introduction

The study of boundary value problems for thin structural problems, such as various thin films in electronic devices, sensors and actuators in smart materials, has received considerable attention in recent years [1–3]. This interest is partly related to the extensive use of smart materials and micro-electro-mechanical systems (MEMS) in various engineering applications. The popular finite element method (FEM) has long been a dominant numerical technique in the simulation of various engineering applications [4,5]. However, as illustrated in Refs. [1,6,7], the *aspect ratio issues* associated with the FEM limit its application to thin-structural problems. To maintain element aspect ratio, a great amount of finite elements should be discretized as the thickness of the structures decreases, which can be arduous, time-consuming and computationally expensive for certain problems. The issue is confirmed indirectly in FEM bibliography, where the relative thickness of 3D thin-walled structures is usually up to the order of 10^{-3}, see Refs. [8,9], for example.

During the past decades, the boundary element method (BEM) has rapidly improved and is nowadays considered as a competing numerical method to the FEM [10–14], thanks to its advantages of semi-analytical nature and boundary-only discretizations. One of the difficulties in the applications of the BEM to structures with thin shapes is the accurate calculation of the *nearly singular integrals* [6,15]. Generally speaking, the integrals are *nearly singular* in the sense that the calculation point is getting towards, but not on, the boundary element. In such case, the integrand, instead of remaining smooth, will develop a sharp peak as the calculation point moves closer to the integration element, thus leading to difficulties

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in accurate evaluation of such integrals [16–18]. Over the past two decades, some considerable effort has been devoted to proposing novel computational techniques to overcome such issues associated with the boundary element analysis. The methods developed so far include, but are not limited to, element subdivision methods [19–21], analytical and semi-analytical methods [22–24] and various nonlinear transformations [1,25–31]. Comprehensive reviews of the earlier work in this regard can be found in Refs. [26,32]. Although impressive results have been obtained, a number of drawbacks still remain and mainly include the fact that some techniques failed to provide accurate results when the thickness of the structure is very small [23,26] while others are tailored to some certain kind of integrals [1,33]. In addition, to our knowledge, very few studies on the calculation of nearly singular integrals arising in general anisotropic BEM formulations have been reported in the BEM community [34].

In this study, we focus on a nonlinear coordinate transformation method, named as the 'sinh' transformation. The main idea of the method is to smooth out the severe oscillation of the nearly singular integrals using a sinh function before the traditional Gaussian quadrature is applied. It is shown that, as illustrated in Refs. [28,35], this approach is accurate, stable and superior to most of existing methods in terms of overall accuracy and stability. The basis of the sinh transformation was published in the early 2000s by Johnston and Elliott [27,36] and were later essentially improved and extended by many other authors. Gu et al. [37] and Lv et al. [38] extended the method to 2D and 3D nearly singular integrals defined on high-order geometry elements (curved surface elements), respectively. In 2013, the sinh transformed BEM developed in [37] was extended to 2D thin structures and applied to the interfacial stress analysis of multi-coating systems. In a more recent study, Gu et al. [39] extended the transformation to nearly singular integrals arising in general 3D anisotropic boundary element analysis.

In this paper, the sinh transformed BEM formulation developed in [39] is extended to 3D thin structural problems associated with anisotropic heat conduction. It is shown that the original 3D anisotropic nearly singular integrals can be transformed into an element-by-element sum of regular integrals, each one expressed in terms of intrinsic (local) coordinates. As a consequence, the actual computation can be performed by using the standard $n \times n$ Gaussian quadrature. For the test problems studied, very promising results have been obtained when the thickness of the structures is in the micro-scales.

A brief outline of the rest of the paper is as follows. The BEM formulations for 3D anisotropic potential problems are described in Section 2. The nearly singular integrals in three dimensions and their numerical implementation are introduced in Section 3. Followed in Section 4, three benchmark numerical examples with thin shapes are well-studied to demonstrate the accuracy and efficiency of the proposed methodology. Finally, the conclusions and remarks are provided in Section 5.

2. BEM formulations for 3D anisotropic heat conduction problems

In this study, we consider anisotropic steady heat conduction applications in the absence of inner heat sources. Hence the function $u(\boldsymbol{x})$, which stands for the temperature distribution in the computational domain Ω, satisfies the following equation [39]

$$k_{11}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_1^2} + k_{22}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_2^2} + k_{33}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_3^2} + 2k_{12}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_1 \partial x_2} + 2k_{13}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_1 \partial x_3} + 2k_{23}\frac{\partial^2 u(\boldsymbol{x})}{\partial x_2 \partial x_3} = 0, \tag{1}$$

subject to the following boundary conditions

$$u(\boldsymbol{x}) = \overline{u}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_{\mathrm{D}} \ (\textit{Dirichlet}\ \text{boundary condition}), \tag{2}$$

$$q(\boldsymbol{x}) = \nabla u(\boldsymbol{x}) \cdot \boldsymbol{m} = \overline{q}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_{\mathrm{N}} \ (\textit{Neumann}\ \text{boundary condition}), \tag{3}$$

where

$$\boldsymbol{m} = (k_{11}n_1 + k_{12}n_2 + k_{13}n_3)\,\boldsymbol{i} + (k_{12}n_1 + k_{22}n_2 + k_{23}n_3)\,\boldsymbol{j} + (k_{13}n_1 + k_{23}n_2 + k_{33}n_3)\,\boldsymbol{k}, \tag{4}$$

is a vector in the direction of the conormal to the boundary, the boundary of Ω is $\partial\Omega \equiv \Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$ which we shall assume to be piecewise smooth, $q(\boldsymbol{x})$ denotes the heat flux at point $\boldsymbol{x}$, $(k_{ij})_{i,j=1,2,3}$ are thermal conductivity coefficients which are assumed to be symmetric and positive-definite so that the partial differential equation (1) is elliptic, the barred quantities $\overline{u}(\boldsymbol{x})$ and $\overline{q}(\boldsymbol{x})$ denote the given values of temperature and heat flux specified along the boundary and $\boldsymbol{n} = (n_1, n_2, n_3)$ indicates the unit outward normal on the boundary points. It is interesting to note that when $k_{11} = k_{22} = k_{33} = 1$ and $k_{12} = k_{13} = k_{23} = 0$, the boundary value problem (1)–(3) reduces to the potential problems in isotropic bodies.

The fundamental solutions of Eq. (1), which is a basic component of all the subsequent analyses, can be written as [38,39]

$$u^*(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi\sqrt{|k_{ij}|}}\frac{1}{r(\boldsymbol{x}, \boldsymbol{y})}, \quad q^*(\boldsymbol{x}, \boldsymbol{y}) = \nabla u^*(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{m}, \tag{5}$$

where

$$r(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{t_{ij}(x_i - y_i)(x_j - y_j)}, \quad (i, j = 1, 2, 3), \tag{6}$$

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