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# Evolutionary subgradient inclusions with nonlinear weakly continuous operators and applications☆

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ABSTRACT

In this paper we consider the first order evolutionary inclusions with nonlinear weakly continuous operators and a multivalued term which involves the Clarke subgradient of a locally Lipschitz function. First, we provide a surjectivity result for stationary inclusion with weakly–weakly upper semicontinuous multifunction. Then, we use this result to prove the existence of solutions to the Rothe sequence and the evolutionary subgradient inclusion. Finally, we apply our results to the non-stationary Navier–Stokes equation with nonmonotone and multivalued frictional boundary conditions.



## 1. Introduction

In this paper we study the first order evolutionary inclusion with a nonlinear weakly continuous, bounded and coercive operator governing the process and a multivalued term involving the Clarke subgradient of a locally Lipschitz function. We formulate the inclusion in the framework of evolution triple of spaces and study it on a finite time interval. This study is closely related to a first order evolutionary hemivariational inequality.

The theory of hemivariational inequalities has been initiated in early 1980s with the pioneering works of Panagiotopoulos, cf. [1,2] and the references therein. The literature on hemivariational inequalities and their applications is today very extensive and still growing. It is well known that such inequalities have often an equivalent formulation as operator subgradient inclusions. Evolutionary hemivariational inequalities have been studied in e.g. [2–6] by using methods based on surjectivity results for various classes of monotone operators, and in e.g. [7–9] by exploiting the Rothe approximation methods.

There are several novelties of the present paper. First, instead of monotonicity, pseudomonotonicity, $L$-pseudomonotonicity, $M$-condition or strong continuity, used in all aforementioned papers, we suppose the weak continuity, boundedness and coercivity conditions of the operator. Weak continuity and coercivity of the operator were used in [10] to study the existence of solution to elliptic equations. We give an extension of this approach to evolutionary subgradient inclusions. Second, our proof uses the Rothe method applied, in contrast to [8,9], to stationary inclusions with weakly continuous operators. Third, as an application, we use weakly continuous operators to provide a simple proof of existence

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of solution to non-stationary Navier–Stokes problem with multivalued frictional boundary condition. Results on solvability of stationary and non-stationary hemivariational inequalities for Navier–Stokes can be found in e.g. [11–16].

The rest of this paper is organized as follows. In the next section, we will briefly recall some definitions and preliminary results. In Section 3, we give a surjectivity result for the stationary inclusion with the multivalued operator which is weakly–weakly upper semicontinuous. In Section 4, we consider the first order evolutionary subgradient inclusion by using the Rothe method. We first prove the existence of solutions for the Rothe problem, and then show a convergence result. Finally, we provide results on uniqueness of solution and on strong convergent of the Rothe sequence. In Section 5 an existence result for the non-stationary Navier–Stokes equation with nonmonotone and multivalued frictional boundary condition is delivered.

## 2. Preliminaries

In this section we introduce preliminary materials and notations, and recall some definitions needed in the sequel.

Let $(X, \|\cdot\|_X)$ be a Banach space. We denote by $X^*$ its dual space, by $w$-$X$ the space $X$ endowed with the weak topology, and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between $X^*$ and $X$. If $X$ is a Hilbert space, we denote by $(\cdot,\cdot)_X$ the inner product of $X$. For $T > 0$, we use the standard Bochner–Lebesgue spaces $L^p(0, T; X)$ with $1 \le p \le \infty$.

We denote by $BV(0, T; X)$ the space of functions of bounded total variation on $(0, T)$. Let $\pi$ denote any finite partition of $(0, T)$ by a family of disjoint subintervals $\sigma_i = (a_i, b_i)$, $i = 1, \ldots, n$ such that $[0, T] = \bigcup_{i=1}^n \overline{\sigma}_i$. Let $\mathcal{I}$ denote the family of all such partitions. Then, for $1 \le r < \infty$, we define a seminorm

$$\|x\|_{BV^r(0,T;X)} = \sup_{\pi\in\mathcal{I}} \left\{ \sum_{\sigma_i\in\pi} \|x(b_i) - x(a_i)\|_X^r \right\}.$$

For Banach spaces $X, Z$ such that $X \subset Z$ we introduce a vector space

$$M^{p,r}(0, T; X, Z) = L^p(0, T; X) \cap BV^r(0, T; Z).$$

Then $M^{p,r}(0, T; X, Z)$ is also a Banach space for $1 \le p, r < \infty$ with the norm given by $\|\cdot\|_{L^p(0,T;X)} + \|\cdot\|_{BV^r(0,T;Z)}$. We need the following result which proof can be found in [9, Proposition 2].

**Lemma 1.** *Let $1 \le p, r < \infty$. Let $X \subset Y \subset Z$ be Banach spaces such that $X$ is reflexive, the embedding $X \subset Y$ is compact and the embedding $Y \subset Z$ is continuous. Then every bounded subset of $M^{p,r}(0, T; X, Z)$ is relatively compact in $L^p(0, T; Y)$.*

Next, we recall some facts on single-valued and multivalued operators in Banach spaces and on Clarke's subgradient which are used in the sequel. More details on these topics can be found in [17–24].

**Definition 2.** Let $X$ be a reflexive Banach space. An operator $F : X \to X^*$ is said to be

(i) weakly continuous, if for any sequence $\{u_n\}_{n\ge1} \subset X$ with $u_n \to u$ weakly in $X$, then $Fu_n \to Fu$ weakly in $X^*$.

(ii) bounded, if there exists a continuous increasing function $M : [0, +\infty) \to [0, +\infty)$ such that

$$\|Fu\|_{X^*} \le M(\|u\|_X) \ \text{ for all } \ u \in X.$$

(iii) coercive, if

$$\lim_{\|u\|_X\to\infty} \frac{\langle Fu, u\rangle_X}{\|u\|_X} = \infty.$$

**Definition 3.** Let $X$ be a reflexive Banach space. A multivalued operator $F : X \to 2^{X^*}$ with closed values is said to be

(i) upper semicontinuous (u.s.c.), if for every open subset $\mathcal{O} \subset X^*$ the "strong inverse image" of $\mathcal{O}$ under $F$ given by $F^+(\mathcal{O}) = \{x \in X \mid F(x) \subset \mathcal{O}\}$ is open in $X$.

(ii) weakly–weakly u.s.c., if for every open subset $\mathcal{O} \subset (w\text{-}X^*)$ the set $F^+(\mathcal{O})$ is open in $w$-$X$.

(iii) closed, if for any $(u_n, u_n^*) \in Gr(F) = \{(u, u^*) \in X \times X^* \mid u^* \in F(x)\}$ with $u_n \to u$ strongly in $X$, $u_n^* \to u^*$ strongly in $X^*$, we have $(u, u^*) \in Gr(F)$.

(iv) weakly–weakly closed, if for any $(u_n, u_n^*) \in Gr(F)$ with $u_n \to u$ weakly in $X$, $u_n^* \to u^*$ weakly in $X^*$, we have $(u, u^*) \in Gr(F)$.

(v) coercive, if

$$\lim_{\|u\|_X\to\infty} \frac{\inf_{u^*\in F(u)}\langle u^*, u\rangle_X}{\|u\|_X} = \infty.$$

It is known, see [22, Theorem 1.1.4] that if $F : X \to 2^{X^*}$ is u.s.c. (weakly–weakly u.s.c.) with closed (weakly closed) values, then $F$ is closed (weakly–weakly closed).

We state below the Kakutani Fixed Point theorem, see [25, Theorem 9.B, p. 452], which originally was formulated by Kakutani in finite dimensional spaces and extended to locally convex spaces by Fan and Glicksberg. Given a multivalued map $F : X \to 2^X$, an element $x \in X$ is called a fixed point of $F$ if and only if $x \in F(x)$.

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