Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A direct algorithm for constrained variational problems in several dimensions

Pablo Pedregal

INEI, U. de Castilla-La Mancha, 13071 Ciudad Real, Spain

ARTICLE INFO

Article history:
Received 10 February 2017
Received in revised form 18 August 2017
Accepted 26 August 2017
Available online xxxx

Keywords:
Obstacle problem
Exponential barrier
Point-wise constraint

ABSTRACT

We introduce a heuristic, practical procedure to take into account, in an easy-to-implement way, point-wise constraints in a variational problem in several dimensions. In addition to showing a convergence result under suitable assumptions, we emphasize the flexibility of the method by using it for an optimal design problem either in conductivity or elasticity, where an optimal mixture of two materials is to be found in a given design domain, under pointwise constraints. We illustrate the performance of the algorithm with some simple test examples.

1. Introduction

The focus of our attention in this contribution is the numerical simulation of the optimal solution of a variational problem under point-wise constraints of the type

$$\text{Minimize in } u(\mathbf{x}) : \quad \int_{\Omega} F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))\, dx$$

under

$$u \in H_0^1(\Omega), \quad \mathbf{G}(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) \le 0 \text{ for a.e. } \mathbf{x} \in \Omega.$$

Though such a general framework is eligible with no special additional effort, to be specific we will place ourselves in a typical scenario to avoid further technical issues. Even more so, we will concentrate in a standard obstacle problem as a paradigm of the kind of situations we would like to deal with

$$\text{Minimize in } u(\mathbf{x}) \in H_0^1(\Omega) : \quad I(u) = \int_{\Omega} \left[\frac{1}{2} |\nabla u(\mathbf{x})|^2 - f(\mathbf{x}) u(\mathbf{x}) \right] d\mathbf{x},$$

subject to

$$u(\mathbf{x}) \ge \psi(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Omega.$$

Ω is a regular domain in $\mathbf{R}^N$, and functions $f(\mathbf{x}) \in L^2(\Omega)$, and $\psi(\mathbf{x})$, the obstacle, continuous in Ω, and with $\psi \le 0$ over $\partial\Omega$, make up the data set of the problem.

The existence of a unique optimal solution for such an optimization problem is not an issue. Indeed, the classical field of variational inequalities (see the book [1,2], but there are many other sources) focuses on optimality conditions for such

E-mail address: pablo.pedregal@uclm.es.

http://dx.doi.org/10.1016/j.camwa.2017.08.039

variational problems. For a variational problem like

$$\text{Minimize in } u(\mathbf{x}) \in \mathbf{K} : \quad \int_\Omega F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))\, d\mathbf{x} \tag{1}$$

where $\mathbf{K}$ is a convex set of competing functions in a Hilbert space like $H_0^1(\Omega)$, we have the following well-known result.

Theorem 1.1. *Let $F(\mathbf{x}, u, \mathbf{U}) : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be convex in the variable $\mathbf{U}$, coercive in the sense*

$$c(|\mathbf{U}|^2 - 1) \le F(\mathbf{x}, u, \mathbf{U}) \le C(1 + |u|^2 + |\mathbf{U}|^2), \quad 0 < c < C,$$

and $\mathbf{K} \subset H_0^1(\Omega)$ convex, and weakly closed. Then (1) admits optimal solutions. If, in addition, $F(\mathbf{x}, u, \mathbf{U})$ is jointly convex in pairs $(u, \mathbf{U})$, and strictly convex in $\mathbf{U}$, the optimal solution is unique.

Typically, as indicated above, the set $\mathbf{K}$ is determined through pointwise or integral inequalities. The study of the contact or coincidence set (where constraints are active) is also a classical topic: its regularity, its free-boundary, . . . , and not just for elliptic problems, but also for parabolic and hyperbolic. We do not pretend to be exhaustive here.

There are various accurate and quite sophisticated schemes for the numerical approximation of such problems. They deal with associated multipliers taking care of constraints, the contact set, and its (free) boundary. Literature on this area is indeed overwhelming. Check the recent contribution [3], and references therein. Because this field (variational inequalities) plays a fundamental role in contact and friction problems in Mechanics, quite a number of references come from it. See [4] for some recent developments. Non-smooth analysis is also an intimately connected field [5].

Our concern is the numerical simulation, through an affordable and flexible method, of such optimal solutions. At this stage, we do not bother about making a full and detailed numerical report, but just to check on various initial academic numerical tests. The elementary method we would like to suggest is based on an accurate way to approximate the optimal solution of regular, smooth, unconstrained variational problems, which is utilized in an iterative way together with an update rule for some auxiliary, additional variables to take care of constraints. To be specific, we will describe the method for the obstacle problem, and, in order to stress its flexibility, we will then move on to a much more complex and demanding situation of an optimal design problem (in conductivity or elasticity) under stress, point-wise constraints. For the one-dimensional case, the method has been examined in [6]. For optimal design problems, it was suggested in [7].

Consider the problem

$$\text{Minimize in } u(\mathbf{x}) \in \mathbf{K} : \quad \int_\Omega \left[\frac{1}{2}|\nabla u(\mathbf{x})|^2 - f(\mathbf{x})u(\mathbf{x})\right] d\mathbf{x} \tag{2}$$

where

$$\mathbf{K} = \{u \in H_0^1(\Omega) : u(\mathbf{x}) \ge \psi(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Omega\},$$

for the two functions f and ψ, as above. Our proposal reads:

(1) Initialization. Take $v_0(\mathbf{x}) \equiv 1$ or any other strictly positive function, and $u_{-1}(\mathbf{x}) \equiv 0$ or any other function in $H_0^1(\Omega)$ (not necessarily in $\mathbf{K}$).
(2) Iterative step.
   (a) Given that we know v_j, and u_{j-1}, approximate the optimal solution u_j of the variational problem

$$\text{Minimize in } w(\mathbf{x}) \in H_0^1(\Omega) : \quad \int_\Omega \left[\frac{1}{2}|\nabla w(\mathbf{x})|^2 - f(\mathbf{x})w(\mathbf{x}) + e^{v_j(\mathbf{x})(\psi(\mathbf{x}) - w(\mathbf{x}))}\right] d\mathbf{x}, \tag{3}$$

   through a typical (variant of a) descent algorithm, starting from u_{j-1}.
   (b) Update rule. If $v_j(\mathbf{x})(\psi(\mathbf{x}) - u_j(\mathbf{x}))$ is sufficiently small

$$\|v_j(\psi - u_j)\|_{L^\infty(\Omega)} \le \epsilon \quad \text{or} \quad \|v_j(\psi - u_j)\|_{L^2(\Omega)} \le \epsilon, \tag{4}$$

   stop, and take u_j as a good approximation of the optimal solution of the obstacle problem. If (4) does not hold, then

$$v_{j+1}(\mathbf{x}) = e^{v_j(\mathbf{x})(\psi(\mathbf{x}) - u_j(\mathbf{x}))} v_j(\mathbf{x}), \tag{5}$$

   and iterate until (4) holds.

Several remarks are worth stating:

- A software package like FreeFem++ [8] looks like an ideal tool to implement such scheme.
- The main computation of the algorithm revolves around the approximation of the optimal solution of (3). It is a regular variational problem, but it is not quadratic, so in practice each optimal solution u_j requires an internal loop to approximate it.

Please cite this article in press as: P. Pedregal, A direct algorithm for constrained variational problems in several dimensions, Computers and Mathematics with Applications (2017), http://dx.doi.org/10.1016/j.camwa.2017.08.039.