ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# A meshless method for solving the time fractional advection–diffusion equation with variable coefficients

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ARTICLE INFO

*Article history:*
Received 9 April 2017
Received in revised form 5 August 2017
Accepted 26 August 2017
Available online xxxx

*Keywords:*
Time fractional advection–diffusion equation
Moving least square (MLS) method
Meshless method

ABSTRACT

In this paper, an efficient and accurate meshless method is proposed for solving the time fractional advection–diffusion equation with variable coefficients which is based on the moving least square (MLS) approximation. In the proposed method, firstly the time fractional derivative is approximated by a finite difference scheme of order $O((\delta t)^{2-\alpha})$, $0 < \alpha \leq 1$ and then the MLS approach is employed to approximate the spatial derivative where time fractional derivative is expressed in the Caputo sense. Also, the validity of the proposed method is investigated in error analysis discussion. The main aim is to show that the meshless method based on the MLS shape functions is highly appropriate for solving fractional partial differential equations (FPDEs) with variable coefficients. The efficiency and accuracy of the proposed method are verified by solving several examples.



## 1. Introduction

Fractional differential equations (FDEs) have attracted increasing attention [1–4] due to their applications in various fields of science and engineering such as many physical and chemical processes, biological systems, etc. Also, fractal time series has been the subject of growing attention as an application of fractional calculus. For example in [5] a fractal time series is taken as the solution of a FDE with a white noise in the domain of stochastic or in [6] fractal time series is used to describe the model of sea level fluctuations. As known, analytic solutions of FDEs mostly cannot be obtained explicitly [7], so the new approaches for finding the numerical solutions of these equations have practical importance. Hence, in recent years several numerical methods have appeared for solving FDEs e.g. [8–20].

We remind that FPDEs are generalizations of classical partial differential equations (PDEs). One important class of FPDEs which widely has been studied is fractional diffusion equation which describes phenomena of anomalous diffusion in transport processes through complex and/or disordered systems including fractal media. In this regard, fractional kinetic equations have been shown to be particularly useful in the context of anomalous slow diffusion. The interested reader is referred to review [21] and references therein.

In this paper, we propose an accurate meshless method based on the MLS approximation for solving the time fractional advection–diffusion equation with variable coefficients in the following form [22]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \beta(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma(x,t)\frac{\partial u(x,t)}{\partial x} + f(x,t), \quad x \in \Omega, \qquad t \geq 0, \tag{1.1}$$

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http://dx.doi.org/10.1016/j.camwa.2017.08.038
0898-1221/

subject to the initial and boundary conditions:

$$\begin{aligned} &u(x,0)=g(x),\\ &u(a,t)=h_1(t), \quad u(b,t)=h_2(t), \end{aligned} \tag{1.2}$$

where $\Omega = [a, b]$, $g(x)$, $h_1(t)$, $h_2(t)$, $\beta(x, t)$, $\gamma(x, t)$ and $f(x, t)$ are given functions, moreover $\beta(x, t) > 0$. Here, $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the fractional derivative of order $0 < \alpha \le 1$ in the Caputo sense of $u(x, t)$ which is defined in [1] by:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-\xi)^{-\alpha} \frac{\partial u(x,\xi)}{\partial \xi} \mathrm{d}\xi, & 0 < \alpha < 1, \\ \dfrac{\partial u(x,t)}{\partial t}, & \alpha = 1. \end{cases} \tag{1.3}$$

In recent years, several meshless methods have been developed for solving different types of ordinary and partial differential equations. These methods have been suggested as alternative numerical tools to overcome difficulty and weaknesses of traditional finite element methods, such as element free Galerkin method, reproducing kernel particle method, local Petrov–Galerkin method, etc. For more details see [23] and references therein.

The MLS method which was first introduced by Shepard [24] and then developed by some researchers, has been used for construction surface and interpolation of scattered data [25–28] and also widely used in the approximation theory. The MLS methodology is an effective approach for approximating an unknown function using a set of disordered data. It consists of a local weighted least square fitting, valid on a small neighborhood of a point and only based on the information provided by its $N$ closet points [29]. In addition, the MLS is accurate and stable for arbitrarily distributed nodes in many problems in computational mechanics [30,31].

Recently, the combination of the MLS method and other methods such as hybrid finite difference [32], finite elements [33], a local meshless method based on the MLS and local radial basis functions [34] and differential quadrature method [35–37] has been proposed.

Many FPDEs are solved using meshless approach based on the radial basis functions e.g. [38–40]. In [30], an implicit meshless approach is described which is based on the MLS approximation using spline weight functions for the numerical simulation of fractional advection–diffusion equation. Also, Tayebi et al. [41] proposed a meshless method based on the MLS approximation for solving two-dimensional variable-order time fractional advection–diffusion equation.

In this paper, we propose the MLS approximation in combination with the finite difference method which is an accurate semi-discrete approach for solving Eq. (1.1), subject to the initial–boundary conditions expressed in Eq. (1.2).

The outline of the paper is as follows: In Section 2, we describe the MLS approximation. Section 3 is devoted to describe the proposed method. In Section 4, numerical analysis and error estimation of the proposed method are investigated. The proposed method is applied for solving some numerical examples in Section 5. Finally, a conclusion is drawn in Section 6.

## 2. The MLS approximation

Suppose that the points $(x_i, u_i)$, $i = 1, \dots, N$ in the domain $\Omega$ are given. The MLS approximation for $u(x)$ can be defined at $x$ by:

$$u^h(x) = \sum_{i=1}^{m} p_i(x) a_i(x) = \mathbf{p}^T(x)\mathbf{a}(x) \tag{2.1}$$

where $\mathbf{p}^T(x) = [p_1(x) \;\; p_2(x) \;\; \dots \;\; p_m(x)]$ is a complete monomials basis of order $m$ and $\mathbf{a}(x)$ is a column vector containing coefficients $a_i(x)$, $i = 1, 2, \dots, m$, which are functions of $x$ and need to be determined. For example, the linear basis is $\mathbf{p}^T(x) = [1 \;\; x]$ and the quadratic basis is $\mathbf{p}^T(x) = [1 \;\; x \;\; x^2]$. The unknown coefficients $a_i(x)$ are determined by minimizing the following weighted discrete $L_2$ norm as:

$$\mathbb{J}(x) = \sum_{i=1}^{N} w_i(x) \big(u^h(x_i, x) - u_i\big)^2 = \sum_{i=1}^{N} w_i(x)(\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i)^2, \tag{2.2}$$

where $w_i(x)$ is the weight function associated with the node $i$ and $N$ denotes the number of nodes in the neighborhood of $x$, where the weight function $w_i(x) > 0$.

The stationarity of $\mathbb{J}$ in Eq. (2.2) with respect to $\mathbf{a}(x)$, i.e. $\frac{\partial \mathbb{J}}{\partial \mathbf{a}} = 0$ yields the following equations:

$$\sum_{i=1}^{N} w_i(x) 2 p_1(x_i)[\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i] = 0,$$

$$\sum_{i=1}^{N} w_i(x) 2 p_2(x_i)[\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i] = 0,$$