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A new matrix splitting preconditioner for generalized saddle point problems

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ABSTRACT

For generalized saddle point problems, we establish a new matrix splitting preconditioner and give the implementing process in detail. The new preconditioner is much easier to be implemented than the modified dimensional split (MDS) preconditioner. The convergence properties of the new splitting iteration method are analyzed. The eigenvalue distribution of the new preconditioned matrix is discussed and an upper bound for the degree of its minimal polynomial is derived. Finally, some numerical examples are carried out to verify the effectiveness and robustness of our preconditioner on generalized saddle point problems discretizing the incompressible Navier–Stokes equations.

1. Introduction

To obtain the numerical solutions of the incompressible Navier–Stokes equations, we often need to solve the following special generalized saddle point problem:

$$\mathcal{A}x = \begin{bmatrix} A_1 & O & B_1^T \\ O & A_2 & B_2^T \\ -B_1 & -B_2 & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ -g \end{bmatrix} \equiv b, \tag{1}$$

where $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $A_2 \in \mathbb{R}^{n_2\times n_2}$ are nonsymmetric positive definite matrices, $B_1 \in \mathbb{R}^{m\times n_1}$ and $B_2 \in \mathbb{R}^{m\times n_2}$ have full row ranks, $C \in \mathbb{R}^{m\times m}$ is a symmetric positive semi-definite matrix and $n_1 + n_2 = n$. If we set

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

the problem (1) can be expressed as the following generalized saddle point problem (2):

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}. \tag{2}$$

It is well known that a block-diagonal preconditioner [1] is ideal for the Stokes equations. Analogously, for the Navier–Stokes equations, Silvester et al. [2] introduced the pressure convection–diffusion (PF) preconditioner and Elman [3] presented the Least-squares commutator (LSC) preconditioner. Recently, there has been tremendous interest in developing some effective preconditioners for this special generalized saddle point problems. Benzi and Guo [4] designed a dimensional split (DS) preconditioner with $C = 0$. When the matrix C is zeros matrix, Benzi et al. [5] further developed an improved

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variant of the DS preconditioner called the relaxed factorization (RDF) preconditioner, and estimated the optimal value of the parameter α based on Fourier analysis. Recently, Gander et al. [6] proposed a new stationary iterative method based on a dimension-wise splitting with selective relaxation (DSSR) and analyzed the optimal choice of the relaxation parameter by using Fourier analysis.

For the general symmetric positive semi-definite nonzero block C, Cao et al. [7] derived a modified DS (MDS) preconditioner. In [8], Cao et al. presented a new relaxed splitting method by constructing a special splitting, and then a modified relaxed splitting (MRS) was considered by Fan and Zhu [9]. However, from the implementing process of the above three preconditioners, we still need to solve the small generalized saddle point problem. Therefore, it consumes much iteration steps and CPU times to obtain the solution of these linear subsystems. To remedy this problem, Ke and Ma [10] constructed a new relaxed splitting preconditioner, numerical experiments show that it demonstrates better convergence behavior than the MRS preconditioner in terms of CPU times.

More recently, Cao et al. [11] derived a simplified Hermitian and skew-Hermitian splitting (SHSS) preconditioner based on the HSS preconditioner. The SHSS preconditioner is much closer to the coefficient matrix $\mathcal{A}$ and much easier to implement than the HSS preconditioner. Using the same technique suggested in [11], Liang and Zhang [12] proposed two new variants of the HSS preconditioner for regularized saddle point problem, Shen et al. [13] presented two improved variants of the deteriorated positive definite and skew-Hermitian splitting (DPSS) preconditioner for generalized saddle point problem, Zhang and Dai [14] designed a new block preconditioner for complex symmetric linear systems. Numerical results show that these preconditioners are effective and robust. Motivated by the ideas in [11–14], we will introduce a new matrix splitting (NMS) preconditioner which is also an improved variant of the MDS preconditioner. Eigenvalue distribution of the preconditioned matrix is investigated and an upper bound of the degree of the minimal polynomial of the preconditioned matrix is derived. Like the preconditioner developed by Ke and Ma [10], we also need not to solve the small generalized saddle point problems. Thus, the new preconditioner is much easier to implement than the HSS, MDS and MRS preconditioners.

This paper is structured as follows. In Section 2, a new matrix splitting preconditioner is proposed and its implementation is given. In Section 3, for the preconditioned matrix, some theoretical results are established. Section 4 is devoted to some numerical examples. Finally, some concluding remarks are made in Section 5.

2. A new matrix splitting preconditioner

In this section, we will develop a new improved variant of the DS preconditioner for the generalized saddle point problems. Inspired by the idea proposed in [11], the coefficient matrix $\mathcal{A}$ of the problem (2) can be split into

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \tag{3}$$

where $\mathcal{A}_1 = \begin{bmatrix} A & O \\ O & O \end{bmatrix}$ and $\mathcal{A}_2 = \begin{bmatrix} O & B^T \\ -B & C \end{bmatrix}$. Analogous to the positive definite and skew-Hermitian splitting (PSS) preconditioner, which was firstly introduced by Bai et al. [14] and used by Pan et al. [15] to solve the saddle point problem, we set

$$\mathcal{P} = \frac{1}{2\alpha}(\alpha I_{n+m} + \mathcal{A}_1)(\alpha I_{n+m} + \mathcal{A}_2). \tag{4}$$

We note that the pre-factor $\frac{1}{2\alpha}$ has no effect on the preconditioned system. Therefore, we can set

$$\mathcal{P}_1 = \frac{1}{\alpha}(\alpha I_{n+m} + \mathcal{A}_1)(\alpha I_{n+m} + \mathcal{A}_2) = \frac{1}{\alpha}\begin{bmatrix} \alpha I_n + A & O \\ O & \alpha I_m \end{bmatrix}\begin{bmatrix} \alpha I_n & B^T \\ -B & \alpha I_m + C \end{bmatrix}. \tag{5}$$

To make the preconditioner $\mathcal{P}_1$ to be as close as possible to the coefficient matrix $\mathcal{A}$, we relax it as follows:

$$\mathcal{P}_2 = \frac{1}{\alpha}\begin{bmatrix} A & O \\ O & \alpha I_m \end{bmatrix}\begin{bmatrix} \alpha I_n & B^T \\ -B & C \end{bmatrix} = \begin{bmatrix} A & \frac{1}{\alpha}AB^T \\ -B & C \end{bmatrix}. \tag{6}$$

From (6), we obtain

$$\mathcal{R}_2 = \mathcal{P}_2 - \mathcal{A} = \begin{bmatrix} O & (\frac{1}{\alpha}A - I_n)B^T \\ O & O \end{bmatrix}. \tag{7}$$

Next, we will give the implementation of the preconditioner $\mathcal{P}_2$. When the preconditioner $\mathcal{P}_2$ is used to the preconditioned Krylov subspace methods, such as GMRES method, we need to solve the following linear subsystems:

$$\frac{1}{\alpha}\begin{bmatrix} A & O \\ O & \alpha I_m \end{bmatrix}\begin{bmatrix} \alpha I_n & B^T \\ -B & C \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \tag{8}$$

where (z_1^T, z_2^T) and (r_1^T, r_2^T) denote the current and generalized residual vectors. From (8), it is easy to derive the implementing process of the preconditioner $\mathcal{P}_2$, which is described in Algorithm 1.

We note that Algorithm 1 is similar to Algorithm 2.1 in [11]. The only difference is in the step 1, i.e., the matrix A is nonsymmetric positive definite in Algorithm 1 and the matrix A is symmetric positive definite in Algorithm 2.1.