Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of the global solution for fractional logarithmic Schrödinger equation

Hongwei Zhang *, Qingying Hu

Department of Mathematics, Henan University of Technology, Zhengzhou 450001, People's Republic of China

ARTICLE INFO

Article history:
Received 30 January 2016
Received in revised form 10 July 2017
Accepted 4 September 2017
Available online xxxx

Keywords:
Fractional logarithmic Schrödinger equation
Initial boundary value problem
Existence of global solution
Family of potential wells

ABSTRACT

In this paper we consider the initial boundary value problem for a class of fractional logarithmic Schrödinger equation. By using the fractional logarithmic Sobolev inequality and introducing a family of potential wells, we give some properties of the family of potential wells and obtain existence of global solution.

1. Introduction

In this paper, we consider the following initial boundary value problem of fractional logarithmic Schrödinger equation

$$iu_t - (-\Delta)^s u + u \log |u|^2 = 0, x \in \Omega, t > 0, \tag{1.1}$$

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), x \in \Omega, \tag{1.3}$$

where $\Omega \subset R^n$, $n \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, the operator $(-\Delta)^s$ with $0 < s \leq 1$ will be defined in Section 2. $u_0(x)$ is a given initial value function and satisfies the boundary condition (1.2).

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in many different contexts, such as, obstacle problems, financial mathematics, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science and water waves. The literature is too wide to attempt a reasonable list of references here, so we refer the reader to the work by Di Nezza, Patalluci and Valdinoci [1,2], where a more extensive bibliography and an introduction to the subject are given. In this paper, we focus on the fractional logarithmic Schrödinger equation.

It is well-known that the fractional Schrödinger equation with power type nonlinearities was derived by Laskin [3,4] by replacing the Brownian motion in the path integral approach with the so called Levy flights. Moreover, the fractional nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + u = f(x,u), x \in R^n,$$

* Corresponding author.
E-mail addresses: whz661@163.com (H.W. Zhang), slxhqy@163.com (Q.Y. Hu).

http://dx.doi.org/10.1016/j.camwa.2017.09.010

has been studied by many authors, we just mention here the earlier works by [5–12] without any attempt to review the references here. The fractional logarithmic Schrödinger equation (1.1) is a generalization of the classical nonlinear Schrödinger equation (NLS) with logarithmic nonlinearity [13]. When $s = 1$, (1.1) becomes the classical logarithmic Schrödinger equation

$$iu_t + \Delta u + u\log|u|^2 = 0. \tag{1.4}$$

The classical logarithmic Schrödinger equation has been ruled out as a fundamental quantum wave equation by very accurate experiments on neutron diffraction, it has been extensively studied in the mathematical and physical literature (see [13–17] and the references therein). Thus, it is natural for us to consider the logarithmic Schrödinger equation with fractional Laplacian. Recently, D'Avenia [18] studied the existence of multiple standing waves solutions to (1.1) by means of nonsmooth critical point theory. They also investigated the Hölder regularity of the weak solutions. Ardila [19] constructed a unique global solution of the associated Cauchy problem of Eq. (1.1) in a suitable functional framework by using a compactness method. They also proved the existence of ground states as minimizers of the action on the Nehari manifold. However, there are few theoretical analyses for the initial boundary value problem of the fractional logarithmic Schrödinger equations (1.1)–(1.3).

In this paper, we aim to study the existence of global weak solution to the problem (1.1)–(1.3) by applying a family of potential wells theory and Galerkin method constructing and estimating the norm of the approximate solutions. The main difficulty in this case is that the methods of potential well in [20–22] will not suitable for the problem (1.1)–(1.3). To handle this difficulty, we followed the idea from reference [23] to introduce the functionals $J(u)$ and $I(u)$ (see Section 2) and use the fractional logarithmic Sobolev inequality [24], then we generalize the family of potential wells introduced by [23] (see also [21,22]) to the problem (1.1)–(1.3) and prove the existence of global solutions to problem (1.1)–(1.3) by using a Galerkin approximation scheme. The idea of the proof for the main result also followed from [23].

This article is organized as follows. Section 2 is concerned with some notations and some lemmas. In Section 3, we introduce the family of potential wells and give some of its properties. In Section 4, we prove the existence of global solutions to problem (1.1)–(1.3) by using a Galerkin approximation scheme combined with the potential well theory.

2. Preliminaries and notations

In this paper, we use standard Lebesgue space $L^p(\Omega)(1 \le p \le \infty)$ by L^p with norm $\|.\|_p$. In the case $p = 2$, we write $\|u\|$ instead of $\|u\|_2$. We also use C to denote a universal positive constant that may take different value in different places.

Let us introduce the definition of fractional Laplacian that we will use in this paper. It is called [25,26] the spectral fractional Laplacian or the "Navier" fractional Laplacian. It follows the idea in [27,28] together with results from [1,29]. The power $(-\Delta)^s$, $0 < s < 1$, of positive Laplace operator $(-\Delta)$, in a bounded domain Ω with zero Dirichlet boundary value, is defined through the spectral decomposition using the powers of the eigenvalues of the original operator [28]. Let $(\lambda_j, w_j)_{j=1}^{\infty}$ be the eigenvalues and eigenfunctions of $(-\Delta)$ in Ω with zero Dirichlet boundary value on $\partial\Omega$, i.e.

$$-\Delta w_j = \lambda_j w_j \text{ in } \Omega, \ w_j = 0 \text{ on } \partial\Omega,$$

normalized by $\|w_j\| = 1$. Then $(\lambda_j^s, w_j)_{j=1}^{\infty}$ are the eigenvalues and eigenfunctions of $(-\Delta)^s$, also with zero Dirichlet boundary value. In fact, the fractional Laplacian $(-\Delta)^s$ is well defined in the space of functions [28]

$$H_0^s(\Omega) = \{u = \Sigma a_j w_j \in L^2(\Omega) : \|u\|_{H_0^s} = (\Sigma a_j^2 \lambda_j^s)^{\frac{1}{2}} < \infty\},$$

and, as a consequence

$$(-\Delta)^s u = \Sigma a_j \lambda_j^s w_j.$$

Note that then $\|u\|_{H_0^s(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|$.

There is another way of defining the fractional Laplacian by using the integral representation in terms of hypersingular kernels, which is called the restricted fractional Laplacian or "Dirichlet" fractional Laplacian in Ω, see e.g. [25,26].

The dual space $H^{-s}(\Omega)$ is defined in the standard way, as well as the inverse operator $(-\Delta)^{-s}$. We have that the embedding $H_0^s(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 < q < \frac{2n}{n-2s}$. In this paper, we have that the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact since $0 < s < 1$ and C_0 is the embedding constant.

We now consider the problem (1.1)–(1.3) in this functional framework. Since the above definition of the fractional Laplacian allows to integrate by parts in the proper spaces, a natural definition of energy solution to problem (1.1)–(1.3) is the following.

We say that $u \in L^\infty(0, T; H_0^s(\Omega))$, $u_t(t) \in L^\infty(0, \infty; L^2)$ is a weak solution of problem (1.1)–(1.3) if the identity

$$i(u_t, v) - ((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v) + (u\log|u|^2, v) = 0,$$

holds for every function $v \in H_0^s(\Omega)$ and a.e. $t \in [0, T]$, and $u(x, 0) = u_0(x)$. Thus, $u(t)$ satisfies the two conservation laws

$$\|u\| = \|u_0\|, E(u(t)) = E(u_0),$$