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A parameter-free dynamic diffusion method for advection–diffusion–reaction problems

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ABSTRACT

In this paper, we present a two-scale finite element formulation, named Dynamic Diffusion (DD), for advection–diffusion–reaction problems. By decomposing the velocity field in coarse and subgrid scales, the latter is used to determine the smallest amount of artificial diffusion to minimize the coarse-scale kinetic energy. This is done locally and dynamically, by imposing some constraints on the resolved scale solution, yielding a parameter-free consistent method. The subgrid scale space is defined by using bubble functions, whose degrees of freedom are locally eliminated in favor of the degrees of freedom that live on the resolved scales. Convergence tests on a two-dimensional example are reported, yielding optimal rates. In addition, numerical experiments show that DD method is robust for a wide scope of application problems.

1. Introduction

Stabilized formulations have been widely used in finite element flow computations in the last forty years. The Streamline-Upwind/Petrov–Galerkin (SUPG) formulations for incompressible and compressible flows [1–3] have become very popular because of their well known desirable features. However, localized oscillations may remain in the neighborhood of sharp layers for nonregular solutions. Many other methodologies were developed to circumvent this difficulty such as the Galerkin/Least-Squares (GLS) [4,5] and the Unusual Stabilized Finite Element Method (USFEM) [6]. A review on a variety of stabilized methods is presented in [7] for the advection-dominated reaction–diffusion equation, in which it is highlighted the critical role of the parameters on stabilization and accuracy. Moreover, since SUPG-like formulations are not monotonicity preserving methods, they all share the common flaw of not precluding spurious oscillations in the neighborhood of sharp layers.

All SUPG-like formulations may be improved by adding discontinuity capturing (DC) terms [8], which ultimately add some sort of artificial dissipation. DC terms are usually nonlinear, since they often depend on the approximate solution. As an example, the Consistent Approximate Upwind Petrov–Galerkin (CAU) [9] is based on a systematic procedure to build an approximate upwind direction towards which stabilization is introduced. This is done locally, at the discrete level, depending on the residual of the approximate solution, and results in two stabilization terms: a linear term that controls the

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solution along the streamlines, and a nonlinear one that provides an artificial diffusion on the remaining directions. Other discontinuity capturing methods presented in the literature are comprehensively studied and compared in [7,10]. Overall, they depend on a suitable design of the stabilization parameters, usually two, for which there is no universal formula. The design depends on the local Peclet number, which in turn depends on the discretization characteristic length, for which again many definitions are possible.

In the mid-nineties, Hughes and co-workers have reformulated stabilized methods within the context of the Variational Multiscale (VMS) framework [11–13]. The underlying idea of the VMS approach is the decomposition of the solution into two-scales: a coarse scale (resolved mesh scale) and the fine or subgrid scales. This allows the variational form of the problem to be decomposed into two problems associated with each of these scales. The next step is to solve, approximate or model the subgrid solution, and then incorporate it into the coarse scale problem, which ultimately yields an enriched problem for the resolved scales. Many well known methods can be placed within this framework such as the residual free bubble method [14], the multiscale finite element method (MFEM) [15], the subgrid scale methods [16–18] and the nonlinear subgrid methods [19–21], to name a few of them. We call the latter group of methods as Dynamic Diffusion (DD) since the dissipation mechanism is locally and dynamically introduced either on all scales or on the subgrid scale, to guarantee stability of the resolved scale solution.

The principle of scale separation is also the focus of the DD method developed here, based on the concepts developed in [21]. The latter is a discontinuous method that also takes into account the effective flux through inter-element edges in order to keep the method consistency. We reformulate this method for a conforming setting and, like in [19–21], we assume a two-level decomposition of the velocity field into the resolved (coarse) and unresolved (subgrid) scales. Unlike the linear subgrid and the nonlinear methods developed in [16–18] and [19,20], respectively, in which the artificial diffusion is introduced only onto the subgrid scales, the present DD method introduces a dynamic artificial diffusion onto both the resolved and unresolved scales. The key issue is to establish the subgrid velocity which is used to determine the smallest amount of artificial diffusion to dissipate the fine-scale kinetic energy such that the residual of the resolved scale solution vanishes. The artificial diffusion, which is introduced onto all scales, is locally and dynamically determined by imposing some restrictions on the resolved scale solution, which ultimately yields a parameter-free method and its consistency property. Preliminary results indicate that this methodology outperforms some discontinuity capturing methods for transport problem [22] and also for compressible flow problems [23].

In this work, DD's method small scale space is defined using bubble functions, whose degrees of freedom are condensed onto the resolved scale degrees of freedom. The methodology leads to a nonlinear scheme, which is solved by an iteration-lagging procedure that utilizes the bubble-enriched Galerkin solution as the correspondent initial approximation. Furthermore, we investigate different ways to represent the characteristic element length and their effect on the method's final accuracy and stability. The remainder of this work is organized as follows. Section 2 describes the proposed Dynamic Diffusion formulation. Numerical experiments are conducted in Section 3 to show the method's behavior for a variety of problems and Section 4 concludes this paper.

2. The dynamic diffusion method

This paper is devoted to the numerical solution of the scalar advection–diffusion–reaction equation modeled by

$$-\epsilon\,\Delta u + \boldsymbol{\beta}\cdot\nabla u + \sigma u = f \qquad \text{in } \Omega, \tag{1}$$

$$u = g \qquad \text{on } \Gamma, \tag{2}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary Γ, u represents the quantity being transported (e.g. temperature, concentration), $\epsilon > 0$ is the (constant) diffusivity, $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$ is the velocity field satisfying $\nabla\cdot\boldsymbol{\beta} = 0$, $\sigma \in L^\infty(\Omega)$ with $\sigma \geq 0$ is the reaction coefficient, $f \in L^2(\Omega)$ is the source term. From now on, we consider $d = 2$ for simplicity, although the whole developed methodology is equally valid for $d = 3$. We adopt the standard notation for Sobolev spaces in which $\|\cdot\|_m$ denotes the standard norm on $H^m(\Omega)$, and $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product (with $H^0(\Omega) = L^2(\Omega)$).

We consider a regular triangulation $\mathcal{T}_h$ of the domain Ω into elements T. The standard Galerkin finite element method for the problem (1)–(2) takes the form:

$$\text{Find } u_h \in V_h \text{ such that } B(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h^0,$$

$$B(u_h, v_h) = \int_\Omega (\epsilon\nabla u_h\cdot\nabla v_h + \boldsymbol{\beta}\cdot\nabla u_h\, v_h + \sigma u_h\, v_h)\, d\Omega, \tag{3}$$

$$(f, v_h) = \int_\Omega f v_h\, d\Omega. \tag{4}$$

Here, without loss of generality, we define the discrete space $V_h = \{w \in H^1(\Omega) \mid w|_T \in \mathbb{P}_1(T),\ \forall T \in \mathcal{T}_h,\ w|_\Gamma = g\}$ and $V_h^0 = V_h \cap H_0^1(\Omega)$ in which $\mathbb{P}_1(T)$ represents the set of first order polynomials in T. The bilinear form $B(\cdot,\cdot)$ is coercive in V_h^0, that is, there exists $\alpha > 0$ such as

$$B(v, v) \geq \alpha\|v\|_1^2, \quad \forall v \in V_h^0, \tag{5}$$

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