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A new condition for blow-up solutions to discrete semilinear heat equations on networks

Soon-Yeong Chung [a], Min-Jun Choi [b,*]

[a] Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 04107, South Korea
[b] Department of Mathematics, Sogang University, Seoul 04107, South Korea

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ABSTRACT

The purpose of this paper is to introduce a new condition

$$(C)\quad \alpha\int_0^u f(s)ds \le uf(u)+\beta u^2+\gamma,\ \ u>0$$

for some $\alpha,\beta,\gamma > 0$ with $0<\beta\le\frac{(\alpha-2)\lambda_0}{2}$, where λ_0 is the first eigenvalue of discrete Laplacian Δ_ω, with which we obtain blow-up solutions to discrete semilinear heat equations

$$\begin{cases} u_t\,(x,t)=\Delta_\omega u\,(x,t)+f(u(x,t)), & (x,t)\in S\times(0,+\infty)\,,\\ u\,(x,t)=0, & (x,t)\in\partial S\times[0,+\infty)\,,\\ u\,(x,0)=u_0\ge 0(nontrivial), & x\in\overline{S}\end{cases}$$

on a discrete network S. In fact, it will be seen that the condition (C) improves the conditions known so far.

0. Introduction

These days, the reaction–diffusion systems have found many applications ranging from chemical and biological phenomena to medicine, genetics, and so on. A typical example of the reaction–diffusion system is an auto-catalytic chemical reaction between several chemicals in which the concentration of each chemical grows (or decays) due to diffusion and difference of concentration (according to Fick's law, for example) and whose phenomena is modeled by the reaction–diffusion system

$$u_t\,(x,t)=\sum_{x\in\overline{S}}[u\,(y,t)-u\,(x,t)]\,\omega\,(x,y)+u^q\,(x,t)\,,\quad (x,t)\in S\times(0,\infty) \tag{1}$$

with some boundary and initial conditions where S is the set of chemicals.

From a similar point of view, we discuss,in this paper, the blow-up property of solutions to the following discrete semilinear heat equations

$$\begin{cases} u_t(x,t)=\Delta_\omega u(x,t)+f(u(x,t)), & (x,t)\in S\times(0,+\infty),\\ u(x,t)=0, & (x,t)\in\partial S\times[0,+\infty),\\ u(x,0)=u_0(x)\ge 0, & x\in\overline{S},\end{cases} \tag{2}$$

which generalizes Eq. (1) and where Δ_ω denotes the discrete Laplacian operator (which will be introduced in Section 1).

* Corresponding author.
E-mail addresses: sychung@sogang.ac.kr (S.-Y. Chung), dudrka2000@sogang.ac.kr (M.-J. Choi).

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The continuous case of this equation has been studied by many authors. For example, in 1973, Levine [1] considered the formally parabolic equations of the form

$$\begin{cases} P\dfrac{du}{dt} = -A(t)u + f(u(t)), \quad t \in [0, +\infty), \\ u(0) = u_0, \end{cases}$$

where P and $A(t)$ are positive linear operators defined on a dense subdomain D of a real or complex Hilbert space H. Here, he first introduced the concavity method and proved that there exists a time T such that

$$\lim_{t\to T^-} \int_0^t \int_\Omega u(x,s)P(u(x,s))dxds = +\infty,$$

under the condition

$$\text{(A)} \quad (2+\epsilon)F(u) \le uf(u), \quad u > 0,$$

for some $\epsilon > 0$ and the initial data u_0 satisfying

$$\frac{1}{2}\int_\Omega u_0(x)\cdot A(0)[u_0(x)]dx + \int_\Omega F(u_0(x))dx > 0,$$

where $F(u) = \int_0^u f(s)ds$.

After this, Philippin and Proytcheva [2] have applied the above method to the equations

$$\begin{cases} u_t = \Delta u + f(u), & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \ge 0, \end{cases} \tag{3}$$

and obtained a blow-up solution, under the condition (A) and the initial data u_0 satisfying

$$-\frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx + \int_\Omega F(u_0(x))dx > 0.$$

Recently, Ding and Hu [3] adopted the condition (A) to get blow-up solutions to the equation

$$(g(u))_t = \nabla\cdot(\rho(|\nabla u|^2)\nabla u) + k(t)f(u)$$

with the nonnegative initial value and the null Dirichlet boundary condition.

Besides, in [4,5] Payne et al. obtained the blow-up solutions to the equations

$$\begin{cases} u_t = \Delta u - g(u), & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial n} = f(u), & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \ge 0, \end{cases} \tag{4}$$

when the Neumann boundary data f satisfies the condition (A).

On the other hand, the condition (A) was relaxed by Bandle and Brunner [6] and has been applied to the equations

$$\begin{cases} u_t = \Delta u + f(x,t,u,\nabla u), & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) \ge 0. \end{cases} \tag{5}$$

In fact, they introduced a condition

$$\text{(B)} \quad (2+\epsilon)F(u) \le uf(u) + \gamma, \quad u > 0$$

and derived the blow-up solutions to Eq. (5), under the condition (B) and the initial data u_0 satisfying

$$-\frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx + \int_\Omega [F(x,u_0) - \gamma]dx > 0,$$

for some $\epsilon > 0$.

Looking into the concavity method more closely, we can see that the proof consists of a series of inequalities with reasoning and the Poincare inequality including the eigenvalue. But the conditions (A) and (B) above are independent of the eigenvalue which depends on the domain. From this observation, we can expect to develop an improved condition which refines (A) or (B), depending on the domain. Being motivated by this point of view, we develop a new condition as follows: for some $\alpha, \beta, \gamma > 0$,

$$\text{(C)} \quad \alpha F(u) \le uf(u) + \beta u^2 + \gamma, \quad u > 0,$$

where $0 < \beta \le \frac{(\alpha-2)\lambda_0}{2}$ and λ_0 is the first eigenvalue of the discrete Laplacian Δ_ω. Here, we note that the term βu^2 is depending on the domain graph.