ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Legendre–Galerkin spectral-element method for the biharmonic equations and its applications

Qingqu Zhuang [a], Lizhen Chen [b,*]

[a] School of Mathematical Sciences, Huaqiao University, 362021 Quanzhou, China
[b] Beijing Computational Science Research Center, Beijing 100193, China

ARTICLE INFO

Article history:
Received 21 December 2015
Received in revised form 13 July 2017
Accepted 24 July 2017
Available online xxxx

Keywords:
Spectral-element method
Legendre–Galerkin approximation
Biharmonic equations
Uniformly loaded plate

ABSTRACT

A spectral-element method based on the Legendre–Galerkin approximation is presented to solve the two-dimensional biharmonic equations. Rigorous error analysis is carried out to establish the convergence of the method. By constructing appropriate basis functions which satisfy the boundary conditions of the differential equations, the discrete variational formulation is reduced to linear system with sparse and symmetric matrices, which can be efficiently solved by a fast Schur-complement method. Accuracy test is provided to confirm the convergence rate of the theoretical results. Finally, the proposed method is applied to calculate the displacement of an elastic plate under a uniform applied load and stream function of zero Reynolds number flow in a driven cavity, the results compare well with established results.

1. Introduction

The biharmonic equation is a fourth-order partial differential equation which arises in areas of continuum mechanics, including bending of thin elastic clamped rectangular plate, equilibrium of an elastic rectangle and flow of very viscous fluid in a rectangular cavity under prescribed walls' motion (see, e.g.,[1]). The numerical solution of this kind of problem by spectral methods has been the subject of numerous studies in recent years. The numerical approaches can be clarified into two major types: direct and mixed. In the first approach, the fourth-order equation is discretized directly by the finite difference or finite element method [2–7]. The Stokes and Navier–Stokes equations involving the stream-function formulation were solved by spectral collocation-type methods in [3]. Based on the Jacobi–Galerkin methods, [7] has presented some efficient direct solvers for general fourth-order equations subject to various boundary conditions. In the mixed approach [8–16], the fourth-order equation is first replaced by a coupled system of two second-order differential equations, and this system is then discretized by the finite difference or finite element method. For example, [13] has proposed a Legendre–Galerkin spectral method for the solution of the biharmonic Dirichlet problem. A fast Schur complement algorithm is presented for computing the piecewise Hermite bicubic orthogonal spline collocation solution of the biharmonic Dirichlet problem on a rectangular region in [14]. Their algorithm is particularly well suited for solving plate bending problems with different kinds of clamped and simply supported boundary conditions. Recently, some splitting schemes by alternating direction implicit method for a fourth-order nonlinear partial differential equation arising in image processing were presented in [16]

Spectral-element method, which combines the advantages of the high accuracy of the spectral method and the geometric flexibility of the finite-element method, is becoming more and more popular [17,18]. Despite the great success of the spectral-element methods applying on second-order elliptic problems, there are few results on spectral-element methods

* Corresponding author.
E-mail addresses: qqzhuang@hqu.edu.cn (Q. Zhuang), lzchen@csrc.ac.cn (L. Chen).

http://dx.doi.org/10.1016/j.camwa.2017.07.039
0898-1221/

Please cite this article in press as: Q. Zhuang, L. Chen, Legendre–Galerkin spectral-element method for the biharmonic equations and its applications, Computers and Mathematics with Applications (2017), http://dx.doi.org/10.1016/j.camwa.2017.07.039.

applying on fourth-order differential equations. The main difficulty of the spectral-element method on fourth-order equations is that we should keep C^1-continuity condition between the elemental faces. To the best of our knowledge, there is only [19] has investigated direct solver by spectral element method for the fourth order problems with mixed inhomogeneous boundary conditions. The aim of this work is to propose a Legendre–Galerkin spectral-element method for the solution of the biharmonic Dirichlet problem subject to homogeneous and nonhomogeneous boundary conditions. A fast Schur-complement algorithm is designed for solving the discrete system. This method was the first to apply in the displacement of an elastic plate under a uniform applied load and stream function of zero Reynolds number flow in a driven cavity. Our work is related to the papers of [4–7,20,21], whose approach is based on the standard variational formulation of the second-order and fourth-order differential equation. On the each sub-domain, the basis functions are tensorial products of appropriate 3-term linear combination of Legendre polynomials on a square. Then we establish the matrix formulations for the discrete variational form, whose mass matrix and stiff matrix are all sparse so it can be solved efficiently. Recently, the same trick have also been successfully used to solve the fractional differential equations [22,23]. Based on the direct method, appropriate basis functions were constructed to imposed the C^1-continuity on the elemental-faces. Similarly idea was used in [24] for unbounded one-dimensional equations.

The outline of this paper is as follows. In Section 2, a Legendre–Galerkin spectral-element method is presented for the two-dimensional homogeneous biharmonic problem and rigorous convergence analysis is carried out. An implementation strategy using carefully chosen modal basis functions is also detailed. Further more, Legendre–Galerkin spectral-element approximation to fourth-order equations with non homogeneous boundary conditions is discussed. In Section 3, we illustrate the spectral accuracy and the efficiency of the algorithm by several examples which including the accuracy test and two applications to the uniformly loaded plate and Stokes flow in a driven cavity. Finally, some concluding remarks are given in Section 4.

2. Biharmonic equation

2.1. Homogeneous biharmonic equation

The two dimensional biharmonic problem is formulated as follows:

Find u, such that,

$$\begin{cases} \Delta^2 u = f, \quad (x,y) \in \Omega = \Lambda \times \Lambda \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where n is the normal vector to $\partial\Omega$, $\Lambda = (-1, 1)$.

In order to imply the Legendre–Galerkin spectral-element method on problem (2.1), we need to further divide Ω into several non-overlapping rectilinear macro-elements:

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}^k, \qquad \Omega^k \cap \Omega^l = \emptyset, \quad \text{for } k \neq l,$$

i.e., we partition Λ in the x direction into K_x parts:

$$\bar{\Lambda} = \bigcup_{k=1}^{K_x} \bar{\Lambda}_x^k, \qquad \Lambda_x^k \cap \Lambda_x^l = \emptyset, \quad \text{for } k \neq l;$$

while partition Λ in the y direction into K_y parts:

$$\bar{\Lambda} = \bigcup_{k=1}^{K_y} \bar{\Lambda}_y^k, \qquad \Lambda_y^k \cap \Lambda_y^l = \emptyset, \quad \text{for } k \neq l.$$

Denote h_k^x and h_k^y the length of Λ_x^k and Λ_y^k respectively. Let $h^x = \max_{1\le k\le K_x} h_k^x$, $h^y = \max_{1\le k\le K_y} h_k^y$, and $h = \max\{h^x, h^y\}$. Moreover, let

$$\mathbb{S}_{\mathcal{N}} := \{\Phi \in L^2(\Omega);\, \Phi_{|\Omega_k} \in \mathbb{P}_N(\Omega_k),\, 1 \le k \le K\} = \mathbb{P}_{N,K_x}(\Lambda) \bigotimes \mathbb{P}_{N,K_y}(\Lambda),$$

where $\mathbb{P}_N(\Omega_k)$ denotes the space of all polynomials of degree at most N with respect to each variable in Ω_k and $\mathcal{N}$ denote the set of discrete parameters (N, K). Now, let

$$Y_{\mathcal{N}} = \mathbb{S}_{\mathcal{N}} \cap H^2(\Omega), \qquad Y_{\mathcal{N}}^0 = \mathbb{S}_{\mathcal{N}} \cap H_0^2(\Omega).$$

Then the Legendre spectral-element approximation to problem (2.1) reads:

Find $u_{\mathcal{N}} \in Y_{\mathcal{N}}^0$ such that

$$d(u_{\mathcal{N}}, v_{\mathcal{N}}) = (f, v_{\mathcal{N}}) \quad \forall v_{\mathcal{N}} \in Y_{\mathcal{N}}^0, \tag{2.2}$$

where $d(u_{\mathcal{N}}, v_{\mathcal{N}}) = (\Delta u_{\mathcal{N}}, \Delta v_{\mathcal{N}})$, and $(\cdot,\cdot)$ represents the inner-product in $L^2(\Omega)$.

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