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An efficient numerical algorithm for the determinant of a cyclic pentadiagonal Toeplitz matrix

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ABSTRACT

In the past few years, a number of numerical and symbolic algorithms for evaluating the determinants of cyclic pentadiagonal matrices have been developed. In this paper, we present a fast numerical algorithm for the determinant of an n-by-n cyclic pentadiagonal matrix with Toeplitz structure. The algorithm is based upon a certain type of matrix reordering and partitioning, and linear transformation. Some numerical examples are provided, and the results are compared with the ones obtained via Matlab built-in function and two existing algorithms. All of the numerical experiments are performed on a computer with the aid of programs written in Matlab.

1. Introduction

We consider the determinant of an n-by-n cyclic pentadiagonal Toeplitz matrix of the form

$$P_n = \begin{bmatrix} d & a & c & & & e & b \\ b & d & a & c & & & e \\ e & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c \\ c & & & e & b & d & a \\ a & c & & & e & b & d \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{1.1}$$

where we assume that $n \geq 5$ and $c \neq 0$. In fact, one may use the arguments given in this paper to obtain similar results when $c = 0$. This type of matrix often appears in boundary value problems (BVPs), quintic spline problems, fluid mechanics, parallel computing, and numerical solution of ordinary and partial differential equations (ODEs and PDEs), especially because the discretization of second-order linear differential equations with periodic boundary conditions, transforming them into finite-difference equations, often results in the cyclic pentadiagonal matrices, see [1–4]. Two important examples of the second-order differential equations that frequently arise in chemical engineering are Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

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and the confluent hypergeometric equation

$$xy'' + (c - x)y' - ay = 0,$$

see [5] for details. Moreover, in paper [6], a sixth-order uniform mesh difference scheme using sextic splines for solving a self-adjoint singularly perturbed two-point boundary-value problem arising in the study of chemical reactor theory, of the form

$$\begin{cases} -\varepsilon u'' + p(x)u = f(x), & p(x) > 0, \\ u(0) = \alpha_0, & u(1) = \alpha_1, \end{cases}$$

is derived. And, the proposed scheme leads to a nearly pentadiagonal Toeplitz matrix which is a special case of the cyclic pentadiagonal Toeplitz matrix (1.1).

It is widely known that a fast and reliable algorithm for computing a square matrix is linked to the problem of obtaining efficient test for the existence of unique solution of the corresponding linear system. Recently, some authors have devised fast computational algorithms for evaluating the cyclic pentadiagonal determinants [7–10]. In this paper, we show that a more efficient numerical algorithm is derived from the use of our framework [9] and the linear transformation given in [11,12].

The cyclic pentadiagonal Toeplitz matrix includes some important classes of matrices such as the pentadiagonal Toeplitz matrix [13–15] and the periodic tridiagonal Toeplitz matrix [16–18]. For related works such as computing the determinants, inverse, and eigenvalues of general (cyclic) pentadiagonal matrices, see [19–24] and the references therein. For recent developments of algorithms for the determinants and permanents of other sparse matrices, see e.g. [25,26].

The rest of this paper is organized as follows. In Section 2, we review the DETQPT algorithm [7] and the DCPT algorithm [8], and provide a comparison of the computational cost between these two algorithms. In Section 3, we present a more efficient algorithm based on a certain type of matrix reordering, matrix partitioning, and linear transformation. Also, the computational complexity of the proposed algorithm is discussed. In Section 4, we report the experimental results of two representative numerical examples for the sake of illustration. Finally, we make some conclusions in Section 5.

2. DETQPT algorithm and DCPT algorithm

In this section, we describe two existing algorithms for the determinants of cyclic pentadiagonal Toeplitz matrices. First, we give the DETQPT algorithm below.

Algorithm 2.1 (DETQPT algorithm [7])

Step 1. Input a, b, c, d, e and n.

Step 2. Set $m_1 = -\frac{a}{c}$, $m_2 = -\frac{d + am_1}{c}$, $m_3 = -\frac{b + dm_1 + am_2}{c}$, $m_4 = -\frac{e + bm_1 + dm_2 + am_3}{c}$.

For $i = 5, 6, \ldots, n-3$ compute

$$m_i = -\frac{em_{i-4} + bm_{i-3} + dm_{i-2} + am_{i-1}}{c},$$

End

Form the following 2-by-2 matrices

$$S = \begin{bmatrix} em_{n-6} + bm_{n-5} + dm_{n-4} + am_{n-3} & em_{n-7} + bm_{n-6} + dm_{n-5} + am_{n-4} \\ em_{n-5} + bm_{n-4} + dm_{n-3} & em_{n-6} + bm_{n-5} + dm_{n-4} \end{bmatrix}.$$

Step 3. Set $r_1 = \frac{1}{c}$, $r_2 = -\frac{a}{c}r_1$, $r_3 = -\frac{dr_1 + ar_2}{c}$, $r_4 = -\frac{br_1 + dr_2 + ar_3}{c}$.

For $i = 5, 6, \ldots, n$ compute

$$r_i = -\frac{er_{i-4} + br_{i-3} + dr_{i-2} + ar_{i-1}}{c},$$

End

Form the following 4-by-4 matrix

$$T = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,n-3} & t_{1,n-2} \\ t_{2,1} & t_{2,2} & t_{2,n-3} & t_{2,n-2} \\ t_{n-3,1} & t_{n-3,2} & t_{n-3,n-3} & t_{n-3,n-2} \\ t_{n-2,1} & t_{n-2,2} & t_{n-2,n-3} & t_{n-2,n-2} \end{bmatrix},$$

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