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Nonlinear damped wave equation: Existence and blow-up

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ABSTRACT

In this paper, we consider the following nonlinear wave equation with variable exponents:

$$u_{tt} - \Delta u + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2},$$

where a, b are positive constants. By using the Faedo–Galerkin method, the existence of a unique weak solution is established under suitable assumptions on the variable exponents m and p. We also prove the finite time blow-up of solutions and give a two-dimension numerical example to illustrate the blow up result.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We consider the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{P}$$

where $a, b \geq 0$ are constants and the exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions on Ω satisfying

$$2 \leq q_1 \leq q(x) \leq q_2 \leq \frac{2n}{n-2}, \quad n \geq 3, \tag{1.1}$$

with

$$q_1 := \text{ess}\inf_{x\in\Omega} q(x), \quad q_2 := \text{ess}\sup_{x\in\Omega} q(x),$$

and the log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \quad \text{with } |x-y| < \delta, \tag{1.2}$$

$A > 0$, $0 < \delta < 1$.

In the case when m, p are constants, local, global existence and long-time behavior have been considered by many authors. For instance, in the absence of the damping term $au_t|u_t|^{m-2}$, the source term $bu|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1,2]). For $b = 0$, it is well-known that the damping term $au_t|u_t|^{m-2}$ assures global existence for arbitrary initial data (see [3,4]).

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The interaction between the damping and the source terms was first considered by Levine (see [2,5]). He discussed the case when $m = 2$ and established the finite-time blow-up for negative initial energy solutions. Georgiev and Todorova [6] generalized the result of Levine to the situation when $m > 2$ by introducing a different technique. Levine et al. [7] extended the previous work to unbounded domains. They proved that any solution with negative initial energy blows up in finite time, if $p > m$. Messaoudi [8] proved that any negative-initial-energy solution blows up in finite time if $p > m$.

In recent years, much attention has been paid to the study of mathematical nonlinear models of hyperbolic, parabolic and elliptic equations with variable exponents of nonlinearity. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. More details on these problems can be found in [9–14]. In fact, there are only few works regarding equations with variable exponents of nonlinearity. Let us mention some of these problems. For instance, Antontsev [16] studied the following problem

$$\begin{cases} u_{tt} = \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u) + \alpha\Delta u_t + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t), & \text{in } \Omega\times(0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}$$

and proved the existence and the blow up of weak solutions with negative initial energy under suitable conditions on the functions a, b, f, p, σ. Alaoui et al. [17] considered the following nonlinear heat equation

$$\begin{cases} u_t = \operatorname{div}\big(|\nabla u|^{m(x)-2}\nabla u\big) = |u|^{p(x)-2}u + f, & \text{in } \Omega\times(0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Under suitable conditions on m and p and for $f = 0$, they showed that any solution with nontrivial initial datum blows up in finite time. They also gave two-dimension numerical examples to illustrate their result. Yunzhu Gao and Wenjie Gao [18] studied a nonlinear viscoelastic equation with variable exponents. They proved the existence of weak solutions by using the Faedo–Galerkin method under suitable assumptions. Autuori et al. [19] looked into a nonlinear Kirchhoff system in the presence of the $\overrightarrow{p}(x,t)$-Laplace operator, a nonlinear force $f(t,x,u)$ and a nonlinear damping term $Q = Q(t,x,u,u_t)$. They established a global nonexistence result under suitable conditions on f, Q, p. We refer the reader to Antontsev [15,16] and Galaktionov [20] for more problems involving the variable-exponent nonlinearities.

Our aim in this work is to prove a local existence theorem and find sufficient conditions on m, p and the initial data for which the blow up takes place. This paper consists of four sections in addition to the introduction. In Section 2, we recall the definitions of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, as well as some of their properties. In Section 3, we prove the local existence of weak solutions for problem (P) by using the method. In Section 4, we state and prove our blow-up result. In Section 5, we give a numerical verification of the blow up result.

2. Preliminaries

In this section, we present some preliminary facts about Lebesgue and Sobolev spaces with variable-exponents (see [17,21–23]). Let $p : \Omega \to [1,\infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We define the Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) := \Big\{u : \Omega \to \mathbb{R};\ \text{measurable in } \Omega :\ \varrho_{p(\cdot)}(\lambda u) < \infty,\ \text{for some } \lambda > 0\Big\},$$

where

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

Equipped with the following Luxembourg-type norm

$$\|u\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx \le 1\Big\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space (see [24]).

We, next, define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega)\ \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$. Furthermore, we set $W_0^{1,p(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Here we note that the space $W_0^{1,p(\cdot)}(\Omega)$ is usually defined in a different way for the variable exponent case. However, both definitions are equivalent under (1.2) (see [24]). The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W^{-1,p'(\cdot)}(\Omega)$, in the same way as the classical Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.