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Conservation laws of partial differential equations: Symmetry, adjoint symmetry and nonlinear self-adjointness

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ABSTRACT

Nonlinear self-adjointness method for constructing conservation laws of partial differential equations (PDEs) is further studied. We show that any adjoint symmetry of PDEs is a differential substitution of nonlinear self-adjointness and vice versa. Consequently, each symmetry of PDEs corresponds to a conservation law via a formula if the system of PDEs is nonlinearly self-adjoint with differential substitution. As a byproduct, we find that the set of differential substitutions includes the set of conservation law multipliers as a subset. The results are illustrated by three typical examples.

1. Introduction

Conservation laws describe the physical properties of the PDEs modeling phenomena. They are used for the study of PDEs such as detecting integrability and linearization, determining constants of motion, finding potentials and constructing nonlocally-related systems, and checking accuracy of numerical solution methods [1,2].

It is well-known that Noether's theorem established a close connection between symmetries and conservation laws for the PDEs possessing a variational structure [1,2]. However, the application of Noether's approach relies on the following two conditions which heavily hinder the construction of conservation laws in such a way:

(1) The PDEs under consideration must be derived from a variational principle, i.e. they are Euler–Lagrange equations.

(2) The used symmetries must leave the variational integral invariant, which means that not each symmetry of the PDEs can generate a conservation law via Noether's theorem. Note that the symmetry stated here and below refers to the generalized symmetry of PDEs if no special notations are added.

Thus many researchers dedicated to develop new approaches to get around the limitations of Noether's theorem [3–7]. In particular, the multiplier method is very effective to construct the conservation laws no matter whether or not the PDEs admit a variational principle. Olver's use of the Euler operator provides a feasible way to find all multipliers in principle [1] while an algorithmic version of this method is the direct construction method where the corresponding local conservation laws are presented through an homotopy integral formula [5–7].

Recently, Ibragimov provides a special method, named by the nonlinear self-adjointness method, to construct some conservation laws of PDEs [8–10]. The two required conditions of this approach are the admitted symmetries and the differential substitutions which convert nonlocal conservation laws to local ones. As for the first requirement, finding the symmetries of the PDEs, there exist a number of well-developed methods and computer algebra programs [11,12]. However, the way to obtain the required differential substitutions is only to use the equivalent identity of the definition involving complicated computations, which even makes us not to get the expected results [8,13,14].

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Therefore in this paper, we show the following two main results:

1. We show that each adjoint symmetry of the PDEs is a differential substitution and vice versa, which gives a positive answer for finding the differential substitutions with a new way. As a byproduct, we find that the set of differential substitutions contains one of the multipliers as a subset.

2. A direct connection among the symmetry, adjoint symmetry and conservation law of the PDEs is expressed by an explicit formula, where the formula only involves differential operation instead of integral operation and thus can be fully implemented on a computer. The above results are exemplified by three illustrated PDEs.

It should be noted that the multiplier method does not require the symmetry information of PDEs but connected with the symmetry and adjoint symmetry [5–7]. On the solution space of the given system of PDEs, multipliers are symmetries provided that its linearized system is self-adjoint, otherwise they are adjoint symmetries and can be obtained by choosing from the set of adjoint symmetries by virtue of the so-called adjoint invariance conditions [6,7]. Quite recently, Anco shows that the general conservation law formula by Ibragimov is equivalent to a standard formula for the action of an infinitesimal symmetry on a conservation law [15–17], what is more, the formula and its earlier version cannot in general produce all admitted conservation laws which are illustrated by some explicit examples [18].

The remainder of the paper is arranged as follows. In Section 2, some related notions and principles are reviewed and the main results are given. In Section 3, three different PDEs are considered to illustrate the connections among symmetry, adjoint symmetry and the differential substitution of nonlinear self-adjointness of PDEs. The last section contains a conclusion of the results.

2. Main results

In this section, we first review some related notions and principles, and then give the main results of the paper.

2.1. Preliminaries

2.1.1. Symmetry, adjoint symmetry and conservation law

Consider a system of m PDEs with rth-order

$$E^\alpha(x,u,u_{(1)},\ldots,u_{(r)})=0,\quad \alpha=1,2,\ldots,m, \tag{1}$$

where $x=(x^1,\ldots,x^n)$ is an independent variable set and $u=(u^1,\ldots,u^m)$ is a dependent variable set, $u_{(i)}$ denotes all the ith x derivatives of u. System (1) is normal if each PDE is expressed in a solved form for some leading derivative of u such that all the other terms in the system contain neither the leading derivative nor its differential consequences [6,7].

On the solution space of the given PDEs, a symmetry is determined by its linearized system while the adjoint symmetry is defined as the solution of the adjoint of the linearized system [1,2].

In particular, the determining system of a symmetry $X_\eta = \eta^i(x,u,u_{(1)},\ldots,u_{(s)})\partial_{u^i}$ is the linearization of system (1) annihilating on its solution space, that is,

$$(\mathscr{L}_E)^\alpha_\rho\eta^\rho=\frac{\partial E^\alpha}{\partial u^\rho}\eta^\rho+\frac{\partial E^\alpha}{\partial u^\rho_{i_1}}D_{i_1}\eta^\rho+\cdots+\frac{\partial E^\alpha}{\partial u^\rho_{i_1\ldots i_r}}D_{i_1}\ldots D_{i_r}\eta^\rho=0 \tag{2}$$

holds for all solutions of system (1). The m-tuple $\eta=(\eta^1,\eta^2,\ldots,\eta^m)$ is called the characteristic of the symmetry. In (2) and below, the summation convention for repeated indices will be used and D_i denotes the total derivative operator with respect to x^i,

$$D_i=\frac{\partial}{\partial x^i}+u^\sigma_i\frac{\partial}{\partial u^\sigma}+u^\sigma_{ij}\frac{\partial}{\partial u^\sigma_j}+\cdots,\quad i=1,2,\ldots,n.$$

The adjoint equations of system (2) are

$$(\mathscr{L}^*_E)^\rho_\alpha\omega_\rho=\omega_\rho\frac{\partial E^\rho}{\partial u^\alpha}-D_{i_1}\left(\omega_\rho\frac{\partial E^\rho}{\partial u^\alpha_{i_1}}\right)+\cdots+(-1)^rD_{i_1}\ldots D_{i_r}\left(\omega_\rho\frac{\partial E^\rho}{\partial u^\alpha_{i_1\ldots i_r}}\right)=0, \tag{3}$$

which are the determining equations for an adjoint symmetry $X_\omega=\omega_\rho(x,u,u_{(1)},\ldots,u_{(r)})\partial_{u^\rho}$ of system (1).

In general, the solutions of the adjoint symmetry determining system (3) are not solutions of the symmetry determining system (2). However, if the linearized system (2) is self-adjoint, then adjoint symmetries are symmetries and system (1) has a variational principle and thus Noether's approach is applicable in this case [1].

Definition 2.1 (*Conservation Law [1]*)**.** A conservation law of system (1) is a divergence expression

$$D_i(C^i)=D_1(C^1)+\cdots+D_n(C^n)=0$$

for all solutions of system (1). If for some $j=1,\ldots,n$, $x^j=t$, then C^t is called the conserved density and the other $C^i(i\neq j)$ are called the spatial fluxes and the pair (C^t,C^i) is called a conserved current.