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Existence and non-existence of nontrivial solutions for Schrödinger systems via Nehari–Pohozaev manifold

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ABSTRACT

This paper is concerned with existence of bound states for a coupled gradient system of nonautonomous Schrödinger equations which arises from studies of nonlinear optics and Bose–Einstein condensates. By using variational methods that are constrained to a Nehari–Pohozaev manifold, we obtain existence of nonnegative and positive bound states solutions. In Particular, we show non-existence result for a minimizing problem.

1. Introduction and main results

In this paper, we study existence and nonexistence of bound states of following elliptic system:

$$\begin{cases} -\Delta u + \lambda_1 u = a(x)F_u(u, v), & x \in \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = a(x)F_v(u, v), & x \in \mathbb{R}^N, \\ u,\ v \geq 0, \quad u,\ v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\lambda_i > 0$, $i = 1, 2$, and $F \in C^2\left((\mathbb{R}^+)^2, \mathbb{R}\right)$, $\mathbb{R}^+ = [0, \infty)$. Following assumptions on a are assumed:

(a_1) $a \in C^2(\mathbb{R}^N, \mathbb{R})$ with $\inf_{x\in\mathbb{R}^N} a(x) > 0$;
(a_2) $\nabla a(x) \cdot x \geq 0$ for all $x \in \mathbb{R}^N$, the inequality is strict in a subset of positive Lebesgue measure;
(a_3) $a(x) + \frac{\nabla a(x)\cdot x}{N} < a_\infty := \lim_{|x|\to\infty} a(x)$ for all $x \in \mathbb{R}^N$;
(a_4) $\nabla a(x) \cdot x + \frac{x\cdot H(x)\cdot x}{N} \geq 0$ for all $x \in \mathbb{R}^N$, where H denotes the Hessian matrix of a.

To state our results, we set $2^* = \frac{2N}{N-2}$ and

$$\mathcal{ND} = \left\{ \begin{array}{l} h \in C^1(\mathbb{R}^+, \mathbb{R}^+),\ h(0) = 0 \text{ and } h'(s) \geq 0 \text{ for } s > 0; \\ \text{there exist constants } p \in (2, 2^*) \text{ and } C_0 > 0 \text{ such that} \\ \qquad 0 < h(s) \leq C_0(1 + s^{p-2}), \quad \forall\, s > 0. \end{array} \right\} \tag{1.2}$$

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Furthermore, we make following hypothesis:

(F_1) there exist functions $f_i \in \mathcal{ND}$, $i = 1, 2, 3$, such that

$$F(u, v) = \int_0^{\sqrt{|uv|}} f_1(s)sds + \int_0^{|u|} f_2(s)sds + \int_0^{|v|} f_3(s)sds,$$

with $f_2(s) + f_3(s) \to \infty$ as $s \to \infty$.

Nonlinear system (1.1) is called a weakly coupled Schrödinger system [1] or a potential system [2]. By (F_1), we see that system (1.1) is fully cooperative (full coupling), which means that the system cannot be reduced to two independent equations, see (3.18). Systems of this type describe the physical phenomena such as the propagation in birefringent optical fibers [3], Kerr-like photo-refractive media in optics and Bose–Einstein condensates [4]. Study of the system is also important for industrial applications in fiber communications systems [5] and all-optical switching devices [6]. We emphasize that coupled Schrödinger systems like (1.1) also arise from the Hartree–Fock theory for the double condensate, that is a binary mixture of Bose–Einstein condensates in two different hyperfine states [7]. For more mathematical and physical background of problem (1.1), we refer the readers to papers [8–10] and the references therein.

Natural space for problem (1.1) is the space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) := E$, which is a Hilbert space equipped with the inner product

$$\langle z, \zeta\rangle = (u, \varphi)_{\lambda_1} + (v, \psi)_{\lambda_2} \ \text{ for any } \ z = (u, v),\ \zeta = (\varphi, \psi) \in E, \tag{1.3}$$

and the corresponding norm

$$\|z\| = \sqrt{\langle z, z\rangle} = \left[\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2\right]^{1/2}, \quad \forall\, z = (u, v) \in E, \tag{1.4}$$

where

$$(u, v)_{\lambda_i} = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda_i uv)dx, \quad \forall\, u,\ v \in H^1(\mathbb{R}^N),\ i = 1, 2. \tag{1.5}$$

Clearly, norm $\|u\|_{\lambda_i} = \sqrt{(u, u)_{\lambda_i}}$ is equivalent to the standard H^1-norm. It follows from Sobolev embedding theorems that the embedding $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}^2)$ is continuous for all $s \in [2, 2^*]$ and $E \hookrightarrow L^s_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ is compact for all $s \in [2, 2^*)$, therefore, there exists constant $\gamma_s > 0$ such that

$$|z|_s \le \gamma_s\|z\|, \quad \forall\, z \in E,\ s \in [2, 2^*], \tag{1.6}$$

where $|\cdot|_s$ stands for the usual $L^s(\mathbb{R}^N, \mathbb{R}^2)$ norm. By (1.4), $\gamma_2 = (\min\{\lambda_1, \lambda_2\})^{-1/2}$.

Since we are looking for nonnegative solutions of (1.1), we take as usual $F(z)$ defined on all $z \in \mathbb{R}^2$, making $F(z) = F(u, v) = 0$ if $u, v < 0$. Define following functional on E,

$$\Phi(z) = \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} a(x)F(z)dx, \quad \forall\, z \in E. \tag{1.7}$$

Our hypotheses imply that $\Phi \in C^1(E, \mathbb{R})$, and a standard argument shows that the solutions of problem (1.1) are critical points of Φ (see [11]). Moreover,

$$\langle \Phi'(z), \zeta\rangle = \langle z, \zeta\rangle - \int_{\mathbb{R}^N} a(x)F_z(z)\cdot\zeta\, dx, \quad \forall\, z,\ \zeta \in E. \tag{1.8}$$

When $a(x) \equiv a_\infty$, we are led to following limiting problem of (1.1),

$$\begin{cases} -\Delta u + \lambda_1 u = a_\infty F_u(u, v), & x \in \mathbb{R}^N,\\ -\Delta v + \lambda_2 v = a_\infty F_v(u, v), & x \in \mathbb{R}^N,\\ u,\ v \ge 0,\ u,\ v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.9}$$

and the associated functional is

$$\Phi_\infty(z) = \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} a_\infty F(z)dx, \quad \forall\, z \in E. \tag{1.10}$$

A solution $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of (1.1) is called a bound state. A bound state such that $(u, v) \ne (0, 0)$ $(u \ge 0, v \ge 0$, respectively) is a nontrivial (nonnegative respectively) bound state. A solution is called a ground state if $(u, v) \ne (0, 0)$ and its energy is minimal among the energy of all the nontrivial bound states of (1.1). A ground state such that $u \ge 0$, $v \ge 0$ $(u > 0$, $v > 0$, resp.) will be called a nonnegative (positive resp.) ground state.

Consider $a(x) \equiv 1$ and $f_i(s) = b_i s^{2q}$, where $q \in (0, \frac{2^*-2}{2})$ and $b_i \in \mathbb{R}$, $i = 1, 2, 3$, then system (1.1) becomes

$$\begin{cases} -\Delta u + \lambda_1 u = b_2|u|^{2q}u + b_1|v|^{q+1}|u|^{q-1}u, & x \in \mathbb{R}^N,\\ -\Delta v + \lambda_2 v = b_3|v|^{2q}v + b_1|u|^{q+1}|v|^{q-1}v, & x \in \mathbb{R}^N, \end{cases} \tag{1.11}$$